Recall: Def: $f:\Omega \to \mathbb{C}$ is holomorphic ($\Omega \subset$ open $\mathbb{C}$) at $z \in \Omega$ if:
$$\lim_{\substack{h\to 0\\ h \in \mathbb{C}}} \frac{f(z+h)-f(z)}{h} \text{ exists } =: f'(z).$$

holo on Ω if true $\forall z \in \Omega$.

Thm: $f = u+iv$ hol'c $\Rightarrow$ (i) $f'(z) = \frac{\partial}{\partial x} f = u_x + i v_x$.
$= \boxed{\frac{\partial}{\partial z} f}$ ← Exercise

where
$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right)$$

& (ii) J_f = Jacobian, $|J| = |f'(z)|^2 = u_x^2 + v_x^2$.

& (iii) (C-R) = (Cauchy-Riemann) $\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$.
$$u_x = v_y \quad \& \quad u_y = -v_x$$

& (iv) $\frac{\partial}{\partial \bar{z}} f = 0$. $\boxed{\text{hol'c} \Rightarrow CR,\ \neg CR \Rightarrow \neg\text{hol'c}}$

E.g. 1: $f = (u,v) = u+iv$, $u = x^2+y^2$, $v = 2xy$.

hol'c?

$u_y = 2y \quad v_x = 2y = -u_y$.

Not hol'c, no C-R.

E.g. 2: $u = x^2 - y^2$, $v = 2xy$ hol'c?.

$$f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + 2ixy.$$

E.g. 3: $u = x^2 - y^2$, $v = -2xy$, $f = \bar{z}^2$

Compute $\frac{\partial}{\partial \bar{z}} f = \frac{1}{2}\left(\frac{\partial}{\partial x}(u+iv) - \frac{1}{i}\frac{\partial}{\partial y}(u+iv)\right)$

$\frac{\partial}{\partial z} = 0$. (anti-hol'c). $= 2x - 2yi = 2\bar{z}$.

E.g. 4: $u = x^2 + y^2$, $v = 0$. $f(z) = |z|^2 = z \cdot \bar{z}$
$= (x+iy)(x-iy)$
$= x^2 + y^2$.

$\boxed{\frac{\partial}{\partial z} f = \bar{z}}$ $\boxed{\frac{\partial}{\partial \bar{z}} f = z.}$ Exercise

$\frac{1}{2}\left(\frac{\partial}{\partial x}(u+\cancel{iv}) + \frac{1}{i}\frac{\partial}{\partial y}(u+\cancel{iv})\right)$.

$= \frac{1}{\cancel{2}}\left(\cancel{2}x + \frac{1}{i}\cancel{2}y\right) = x - iy = \bar{z}$

Converse Thm: If u & v are $\mathbb{R}^2$-diff'ble near z & C-R. $\Longleftrightarrow$ $\underset{u+iv}{f}$ is hol'c at z.

u is $\mathbb{R}^2$-diff'ble: $u(x+h_1, y+h_2)$

$h = h_1 + ih_2$, $|h| \to 0$

$$= u(x,y) + \underbrace{\begin{bmatrix} u_x & u_y \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix}}_{u_x \cdot h_1 + u_y \cdot h_2.} + o(|h|)$$

same for v.

pf: look at $f(z+h) - f(z) = u(z+h) - u(z) + i(v(z+h) - v(z))$.

$$= u_x \cdot h_1 + u_y \cdot h_2 + i(v_x \cdot h_1 + v_y \cdot h_2) + o(|h|).$$

$$= u_x \cdot h_1 - v_x h_2 + i(v_x \cdot h_1 + u_x \cdot h_2) + o(|h|).$$

$$= (u_x + iv_x)h_1 + ih_2(u_x + iv_x) + o(|h|).$$

$$= (u_x + iv_x)\underbrace{(h_1 + ih_2)}_{h} + o(|h|).$$

$$\frac{f(z+h) - f(z)}{h} = \frac{\partial f}{\partial x} + o(1).$$

'' $f'(z)$

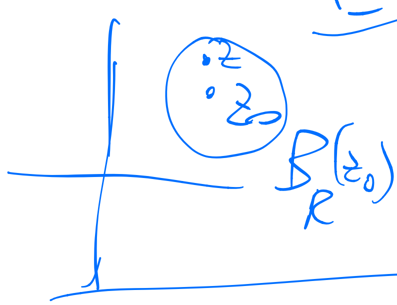
Caveat!! Not enough for u_x, u_y, v_x, v_y exist & satisfy CR for holomorphicity.

E.g.: $f = \dfrac{\bar{z}^5}{|z|^4}$ ✗ play $\boxed{f = \dfrac{\bar{z}^2}{z}}$.

Recall: Def $f : \Omega \to \mathbb{C}$ is "analytic near $z_0 \in \Omega$"

if $\exists\ \infty \geq R > 0$ &

$$\underbrace{\sum_{n\geq 0} a_n (z - z_0)^n}_{= f(z)} \quad \text{conv abs} \quad z \in \overset{\text{open}}{B_R(z_0)}$$



$\boxed{z_0 = 0}$.

E.g.:

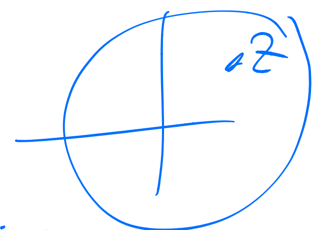
- $\displaystyle\sum_{n\geq 0} \frac{z^n}{n!} = \exp(z)$. $\boxed{R = \infty}$.
- $\displaystyle\sum \frac{z^{2n}}{(2n)!}(-1)^n = \cos z = \frac{e^{iz} + e^{-iz}}{2}$
- $\displaystyle\sum (-1)^n \frac{z^{2n+1}}{(2n+1)!} = \sin z = \frac{e^{iz} - e^{-iz}}{2i}$.

$|a_n| \leq \frac{1}{n!}$

• $\sum z^n = \frac{1}{1-z}$, $|z| < R = 1$. $\boxed{a_n = 1.}$

$\Rightarrow$ • $a_n = 2^n$. $\sum 2^n z^n = \frac{1}{1-2z}$, $|z| < \boxed{\frac{1}{2} = R}$

• $a_n = n!$ $\sum n! z^n$.



$R = 0$.

$\Rightarrow$ for $n > |z|$,

$n! z^n \to \infty$. terms $\not\to 0$. $\Rightarrow$ diverges

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What happens in general.

Thm (Hadamard): Let $\sum a_n z^n$ be a power series. $\exists\, 0 \le R \le \infty$ s.t. (i) $\forall\, |z| < R$, series conv abs.,

(ii) $\forall\, |z| > R$ series diverges.

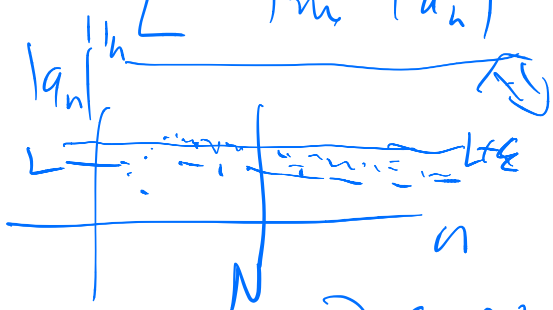
& $R = \frac{1}{L}$, where $L = \limsup |a_n|^{1/n}$.

E.g. $a_n = \frac{1}{n!}$, $\left|\frac{1}{n!}\right|^{1/n}$, $n! \approx n^n e^{-n}\sqrt{n}$

$L = 0 \Leftarrow R = \infty$.

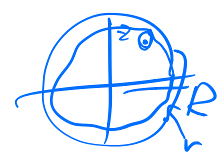
pf (i): Assume $0 < R < \infty$. ($R = 0$, ∞ exercise)

$L = \overline{\lim}\, |a_n|^{1/n}$. Let $|z| < R$.



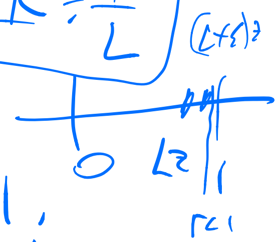
$\forall \epsilon > 0\ \exists N\ \forall n > N$

$|a_n|^{1/n} < L + \epsilon$.



$\exists \epsilon$ s.t. $\boxed{|z| \le r < R = \frac{1}{L}}$

$\boxed{L|z| < 1}$



Let ϵ be small enough $(L+\epsilon)|z| \le r < 1$.

Look at tail $\sum_{n>N} |a_n||z|^n \le \sum_{n>N} (L+\epsilon)^n |z|^n \le \sum_{n \ge N} r^n < \infty$.

$\boxed{\text{Added Exercise to be handed in}}$

Exercise Prove (ii)

Thm: Analytic ⟹ holo.

If $f = \sum a_n z^n$ has $R>0$,

then f is holo on $|z|<R$,

& $f'(z) = \sum n a_n z^{n-1}$

which has same radius R of

abs convergence. $a_n' = (n+1)a_{n+1}$.

Key ideas: Know f is

Analysis Let $S_N(z) = \sum_{n \le N} a_n z^n$

& $E_N(z) = \sum_{n > N} a_n z^n$.

Claim: Let $g(z) = \sum n a_n z^{n-1}$

$n^{1/n} = \ell$, $\log \ell = \frac{1}{n} \cdot \log n \to 0$,

has abs conv for $|z| < R$,

Want: $\left| \underbrace{\frac{f(z+h) - f(z)}{h}} - \underbrace{g(z)} \right| \to 0$,

3ε argument.

$$f(z) = S_N + E_N$$

$$\leq \underbrace{\left| \frac{S_N(z+h) - S_N(z)}{h} - S_N' \right|}_{< \varepsilon}$$

$$+ \left| S_N' - g \right|_{< \varepsilon}$$

$$+ \left| \frac{E_N(z+h) - E_N(z)}{h} \right|_{< \varepsilon}$$

$$\frac{\partial}{\partial \bar{z}} f = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{\bar{h}}$$

$$x = \frac{z + \bar{z}}{2}, \quad y = \frac{z - \bar{z}}{2i}$$

$$u = x^2 + y^2, \quad v = 0$$

$$f = \left(\frac{z + \bar{z}}{2}\right)^2 + \left(\frac{z - \bar{z}}{2i}\right)^2$$

$$= z \cdot \bar{z}$$

In[1]:= `z = x + I y; zbar = x - I y`

Out[1]= $x - i\, y$

In[2]:= `f = (z^3 / zbar^2 // Expand)`

Out[2]= $$\frac{x^3}{(x - i\, y)^2} + \frac{3\, i\, x^2\, y}{(x - i\, y)^2} - \frac{3\, x\, y^2}{(x - i\, y)^2} - \frac{i\, y^3}{(x - i\, y)^2}$$

In[3]:=
```
u = FullSimplify[(f + Conjugate[f]) / 2,
  Assumptions → (x ∈ Reals && y ∈ Reals && x ≠ 0 && y ≠ 0)]
```

Out[3]= $$\frac{x^5 - 10\, x^3\, y^2 + 5\, x\, y^4}{\left(x^2 + y^2\right)^2}$$

In[4]:=
```
v = FullSimplify[(f - Conjugate[f]) / (2 I),
  Assumptions → (x ∈ Reals && y ∈ Reals && x ≠ 0 && y ≠ 0)]
```

Out[4]= $$\frac{5\, x^4\, y - 10\, x^2\, y^3 + y^5}{\left(x^2 + y^2\right)^2}$$

In[5]:=
```
ux = D[u, x]
uy = D[u, y]
vx = D[v, x]
vy = D[v, y]
```

Out[5]= $$\frac{5\, x^4 - 30\, x^2\, y^2 + 5\, y^4}{\left(x^2 + y^2\right)^2} - \frac{4\, x\, \left(x^5 - 10\, x^3\, y^2 + 5\, x\, y^4\right)}{\left(x^2 + y^2\right)^3}$$

Out[6]= $$\frac{-20\, x^3\, y + 20\, x\, y^3}{\left(x^2 + y^2\right)^2} - \frac{4\, y\, \left(x^5 - 10\, x^3\, y^2 + 5\, x\, y^4\right)}{\left(x^2 + y^2\right)^3}$$

Out[7]= $$\frac{20\, x^3\, y - 20\, x\, y^3}{\left(x^2 + y^2\right)^2} - \frac{4\, x\, \left(5\, x^4\, y - 10\, x^2\, y^3 + y^5\right)}{\left(x^2 + y^2\right)^3}$$

Out[8]= $$\frac{5\, x^4 - 30\, x^2\, y^2 + 5\, y^4}{\left(x^2 + y^2\right)^2} - \frac{4\, y\, \left(5\, x^4\, y - 10\, x^2\, y^3 + y^5\right)}{\left(x^2 + y^2\right)^3}$$

In[9]:= `Limit[ux /. y → 0, x → 0] == Limit[vy /. x → 0, y → 0]`

Out[9]= True

In[10]:= `Limit[vx /. y → 0, x → 0] == -Limit[uy /. x → 0, y → 0]`

Out[10]= True

- However, $f(z) = u(x, y) + i\, v(x, y)$, is not holomorphic! So what’s going on?

- If we view the variable in which we are *not* taking the derivative of u as already “fixed” at our chosen value (here, $y = 0$) before we take that derivative, we get a different value than if we had come in from some other direction.
- That is, the derivative of the partially-applied function u(x,0) with respect to x, evaluated at $x = 0$, (which is what the above Cauchy Riemann equations are apparently “satisfied” by) is not the limit of the derivative of $u(x, y)$ with respect to x, at $x = 0$, as we approach $y = 0$,.

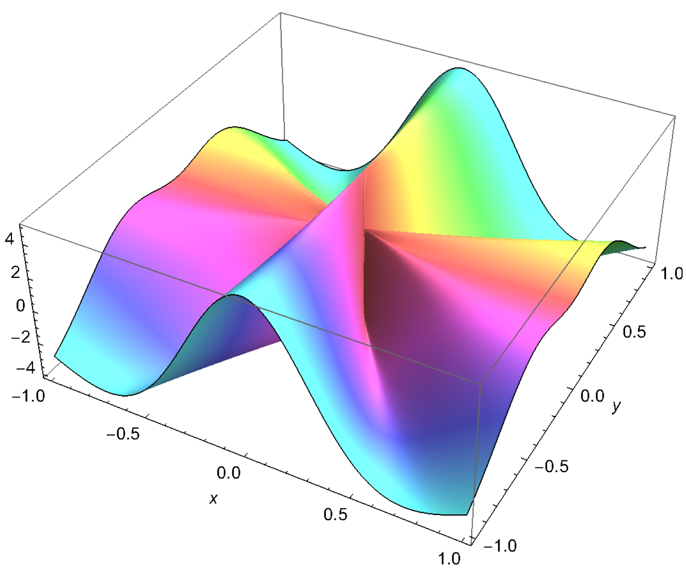
Out[∘]//TraditionalForm=

$$\left[\frac{\partial}{\partial x} u(x, 0)\right]_{x=0} \neq \lim_{y \to 0}\left(\left[\frac{\partial}{\partial x} u(x, y)\right]_{x=0}\right)$$

- This is a plot of $u_x(x, y) := \frac{\partial}{\partial x} u(x, y)$, with slices of constant y given the same color.

In[11]:=
```
Plot3D[ux, {x, -1, 1}, {y, -1, 1}, PlotPoints → 200,
 AxesLabel → Automatic, ColorFunction → (Hue[#2 + 0.5, 0.6, 1] &), Mesh → False]
```

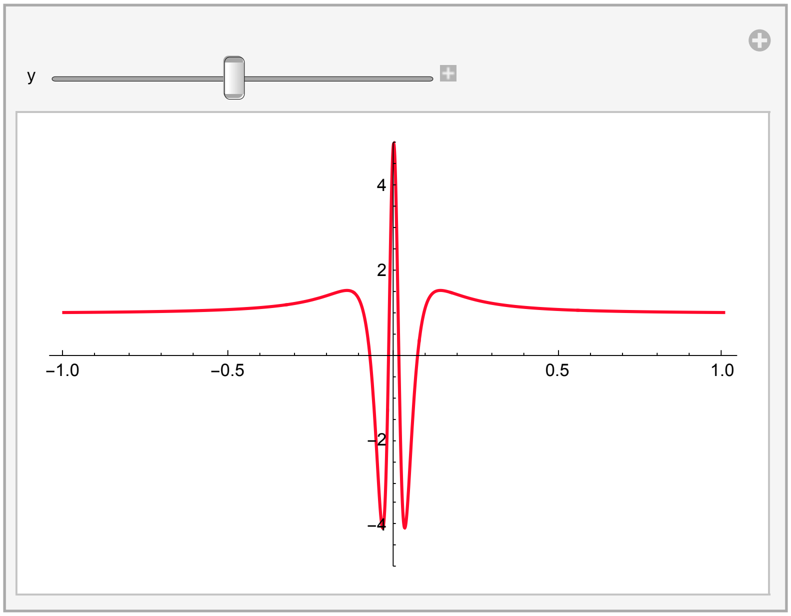
Out[11]=



- The following is a vertical slice of the above plot at the corresponding color; that is, it’s a plot of $u_x(x, y)$ with respect to x, with y determined by the slider position. The color reflects where we are in the above plot.

In[12]:=
```
With[{ux = ux}, Manipulate[Plot[ux, {x, -1, 1}, PlotRange → {-5, 5},
   PlotStyle → {Hue[y / 2, 1 - 0.8 Abs[y], 1]}], {y, -1, 1}]]
```

Out[12]=



- Note the behavior at $y = 0$,: $u_x(x, y)$ appears to become 1 at $x = 0$ instead of 5! This equals $v_y(x, y)$ evaluated the analogous way: differentiating with respect to y after setting $x = 0$.