New Topic: Hypothesis testing.

Hypothesis Fact either true or false.

E.g.: This coin is fair. I.e. X = Bernoulli with θ. Fair is if $\theta = 1/2$.

Innocent until proven guilty.

Null hypothesis H_0: "nothing to see here".
"Coin is fair" "this drug does not improve outcomes".

Test H_0 against alternative hypothesis H_1: not necessarily $\neg H_0$.

E.g.: Pharma manufacturers. Proportion of population that gets a disease when exposed to a virus is 90%. Test hypothesis that their drug decreases proportion of diseased to 60%.

~~H_0~~ Population: People exposed to the virus & given drug.

H_0: the drug does not help. X_i = Bernoulli $\theta = 0.9$.

H_1: alternative: X_i = Bernoulli with $\theta = 0.6$.

Test: give $n=20$ people drug. Reject H_0 if k = # get disease is $\underline{k \le 14}$.

$\boxed{\text{Q: how good is this test?}}$

Def: Critical region/value: set of outcomes for which we reject H_0.

	H_0 true	H_0 false
"accept" H_0 / fail to reject	✓	
reject H_0		✓

Type II error: accept H_0 but H_1 is true.

Type I error: reject H_0, but H_0 = true

$\mathbb{P}(\text{Type I error}) = \alpha, \quad \mathbb{P}(\text{Type II error}) = \beta.$

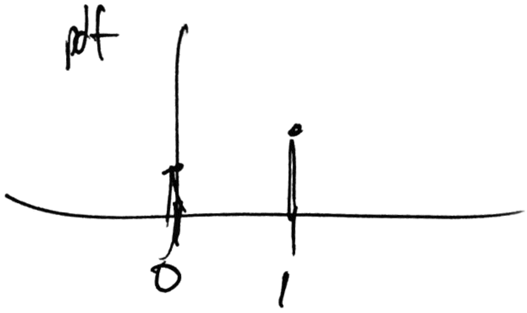
$$\alpha = \mathbb{P}(\text{Type I error}) = \mathbb{P}(H_0 \text{ is true \& reject } H_0)$$
$$= \mathbb{P}(\theta = 0.9 \;\&\; n\bar{X} \le 14) = \sum_{k=0}^{14} \binom{20}{k} 0.9^k (.1)^{20-k} = 0.011.$$

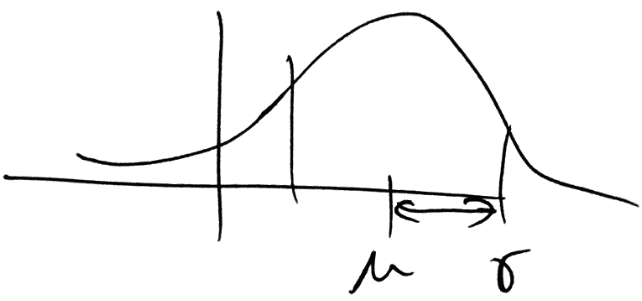
$\boxed{\alpha = 1.1\%}$ = "size of critical region" = "level of significance".

$$\beta = \mathbb{P}(\text{Type II error}) = \mathbb{P}(H_1 \text{ is true \& accept } H_0)$$
$$= \mathbb{P}(\theta = 0.6 \;\&\; n\bar{X} \ge 15) = \sum_{k=15}^{20} \binom{20}{k} 0.6^k (.4)^{20-k} = 0.1256.$$
$$= 12\%.$$

E.g.: **Brachistochrone**: What is the path of shortest time for a ball to roll from A to B.

(variational analysis). (only Gravity).

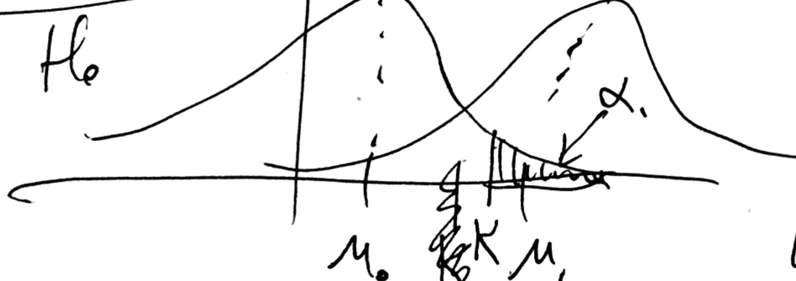




E.g.: normal population $X_1, \dots, X_n$, var $\sigma^2 = 1$. $\mu_1 > \mu_0$

Test $H_0: \mu = \mu_0$ against $H_1: \mu = \mu_1$. ← two simple hypotheses.

When the hypothesis completely determines underlying pdf, that hypothesis is called simple. Otherwise hypothesis is composite, e.g. $\theta < 0.4$.



Q: Find a value of K s.t. $\bar{x} > K$ ← critical region has level of significance $\alpha = 0.05$

$\alpha = \mathbb{P}(\text{Type I error}) = \mathbb{P}(H_0 \text{ rejected \& true})$

$= \mathbb{P}(\mu = \mu_0 \ \& \ \bar{x} > K)$ $\bar{X} \sim N(\mu_0, \sigma^2 = \frac{1}{n})$.

$$P(\bar{X} > K) = \int_{K}^{\infty} \frac{1}{\sqrt{2\pi/n}} e^{-\frac{(z-\mu_0)^2}{2/n}} \, dz.$$

$$e^{-\frac{(z-\mu)^2}{2\sigma^2}}.$$

Need to convert to standard normal so we can "look up" z_α: $\displaystyle\int_{z_\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz = \alpha.$

Let $\dfrac{u^2}{2} = \dfrac{(z-\mu_0)^2}{2/n}$. $\quad u = (z-\mu_0)\sqrt{n}$.

$dz = \frac{1}{\sqrt{n}} du$. $\quad \dfrac{u}{\sqrt{n}} + \mu_0 = z$.

$$\int_{(K-\mu_0)\sqrt{n}}^{\infty} \frac{1}{\sqrt{2\pi/\cancel{n}}} e^{-\frac{u^2}{2}} \frac{1}{\cancel{\sqrt{n}}} du = \alpha.$$

$(K-\mu_0)\sqrt{n} \longrightarrow (K-\mu_0)\sqrt{n} = z_\alpha$. $\quad 1.645$.

$$\Rightarrow \boxed{K = \mu_0 + \frac{z_\alpha}{\sqrt{n}}}.$$

Part 2: Is H_1 "good" i.e. what is $\beta = P(\text{Type II error})$.

Def. When testing H_0, the alternative H_1 has power $1-\beta$.

I.e. Find power of H_1.

$$\beta = \mathbb{P}(\text{Type II error}) = \mathbb{P}(\mu = \mu_1 \ \& \ \underset{\sim}{\bar{X}} < K).$$

$\bar{X} \sim N(\mu_1, \frac{1}{n})$.

$$= \mathbb{P}\left(\underbrace{\frac{\bar{X} - \mu_1}{\sqrt{1/n}}}_{\text{standard normal.}} < \frac{K - \mu_1}{\sqrt{1/n}} \right)$$

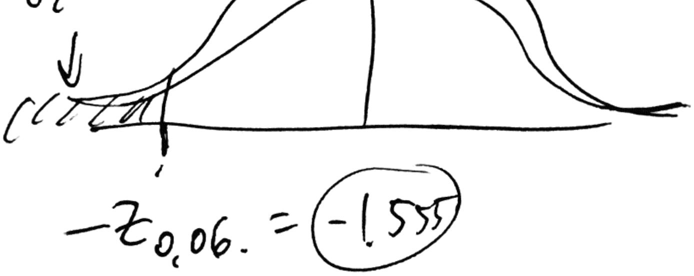
$$= \mathbb{P}\left(Z < \frac{\mu_0 + \frac{z_\alpha}{\sqrt{n}} - \mu_1}{\sqrt{1/n}} \right).$$

Recap: K was chosen to give a critical region of level of significance α. So $K = (\alpha, \mu_0)$. Then the power of the test is $1-\beta$, where

$$\beta = -\int_{(\mu_0 - \mu_1)\sqrt{n} + z_\alpha}^{-\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}}\, du.$$

Q: How many n to test if we want $\beta \le 0.06$ for $\mu_0 = 10$, $\mu_1 = 11$. ($\alpha = 0.05$).

0.06



$-z_{0.06} = -1.555$

$$(\mu_0 - \mu_1)\sqrt{n} + z_\alpha \le -1.555$$

$$-\sqrt{n} + 1.645 \le -1.555$$

$$3.2 \le \sqrt{n}$$

$$10.24 \le \boxed{n = 11}$$