Recall: T_F total variation

$$= \sup_{\substack{P \subset [a,b] \\ a=t_0<\dots<t_N=b}} \sum_{i=1}^{N} |F(t_i) - F(t_{i-1})| .$$

F is of **bdd var** if $T_F < \infty$.

(For $F = \gamma$, T_γ = length $\Leftrightarrow$ rectifiable).

- $T_F = P_F + N_F$, $F(x) = F(a) + P_F(a,x) - N_F(a,x)$.

pos var (P_F), neg var (N_F)

Thm: F bdd var $\Leftrightarrow$ $F = F_1 - F_2$, with F_1, F_2 increasing, bdd.

Rising Sun Lemma: (Riesz) Let $G : [a,b] \to \mathbb{R}$ continuous,

Let $E = \{a < x < b \mid \exists h > 0 : G(x+h) > G(x)\}$

Then E = open, $E = \bigsqcup_i (a_i, b_i)$ &

$G(a_i) \le G(b_i)$. ($=$ except at $a_i = a$)

<u>Goal</u>: F bdd var $\Rightarrow$ F' exists a.e.

Let F be increasing, bdd, define Dini derivatives:

$$D^+F(x) = \limsup_{\substack{h \to 0^+ \\ (h>0)}} \frac{F(x+h) - F(x)}{h}.$$

$D_+ \le D^+$

$$D_+F(x) = \liminf_{h \to 0^+} \text{''}$$

$D_- \le D^-$

$$D^-F(x) = \limsup_{h \to 0^-} \text{''}$$

$$D_-F(x) = \liminf_{h \to 0^-} \text{''}$$

F increasing, bdd, & cont.

Step 1. Claim: $\{D^+F < \infty\}$ a.e.

Pf: (Exercise: D^+F measurable).

$\Rightarrow \forall n \geq 1$, $E_n := \{x \mid D^+F(x) > n\} \in \mathcal{M}$.

$\{x \mid D^+F = \infty\} = \bigcap E_n$.

Apply Rising Sun to $G(x) = F(x) - nx$.

$E = \{x \mid \exists h > 0 : G(x+h) > G(x)\}$.

$F(x+h) - n(x+h) > F(x) - nx$

$\frac{F(x+h) - F(x)}{h} > n$

$E_n \subseteq E = \bigsqcup (a_i, b_i)$

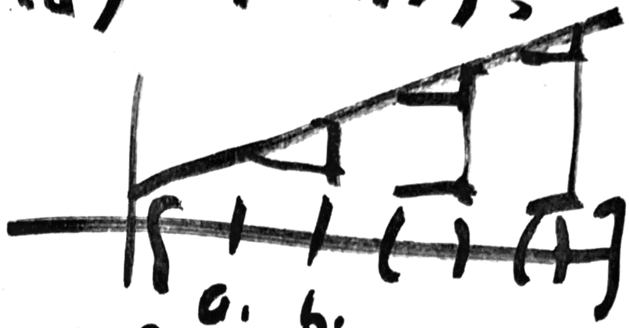
& $G(a_i) \leq G(b_i)$

$F(a_i) - na_i \leq F(b_i) - nb_i \Rightarrow F(b_i) - F(a_i) \geq n(b_i - a_i)$.

So $m(E_n) \leq \sum_i (b_i - a_i) \leq \frac{1}{n} \sum_i F(b_i) - F(a_i)$

$m(E_n) \le \frac{1}{n}\left(F(b) - F(a)\right)$.

$\Rightarrow m\{x \mid D^+F = \infty\}$



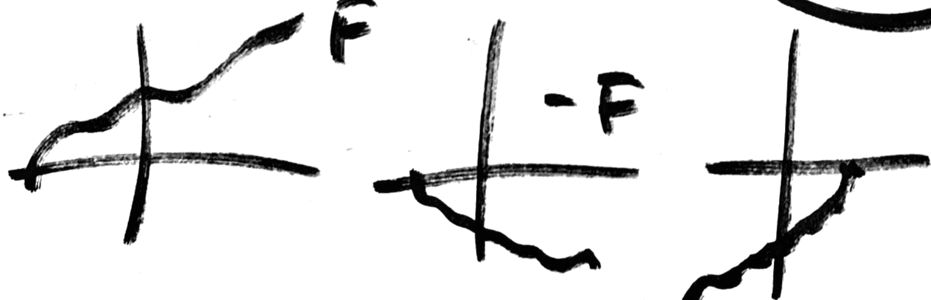
$\le \frac{1}{n}\left(F(b) - F(a)\right) \to 0$ as $n \to \infty$.

$\Rightarrow D^+F < \infty$ a.e.

<u>Step 2</u>: <u>Claim</u>: $D^+F \le D_-F$ a.e.

<u>pf (Step 2 $\Rightarrow$ Thm)</u>: $D^+F \le D_-F \le D^-F$

Apply Step 2 to $-F(-x)$ $\to$ $\le D_+F \le D^+F$.
$< \infty$ a.e.



So F' exists a.e.

<u>pf (Step 2)</u>: $\{x \mid D^+F > D_-F\} =$

$$= \bigcup_{R, r \in \mathbb{Q},\, R > r > 0} \underbrace{\{D^+F > R \,\&\, D_-F < r\}}_{E_{Rr}} \leftarrow \text{Want } m = 0.$$

Assume $m(E_{R,r}) > 0$. Measurable so $\exists \mathcal{O}$ open $\mathcal{O} \supset E_{R,r}$

$\mathcal{O} \subset [a,b]$ with $m(\mathcal{O}) < m(E_{R,r})\frac{R}{r}$.

$= \bigsqcup I_n$. On each I_n, apply Rising Sun to: $G(x) = -F(-x) + rx \big|_{-I_n}$. Then

open $= \{x \in -I_n \mid \exists h > 0 : G(x+h) > G(x)\}$.

~~open $\{x \in I_n \mid \exists h>0 \ -F(-x-h) + rx + rh > -F(-x) + rx\}$~~

~~$\exists h>0 \ F(x+h) - F(x) < r$~~

$\hookrightarrow -F(-(x+h)) + r(x+h) > -F(-x) + rx$.

$-\{x \in -I_n : \exists h>0 \ r > \frac{F(-x-h) - F(-x)}{h}\}$.

$= \bigsqcup (a_j, b_j)$ & $G(-b_j) \le G(-a_j)$.

$G(-b_j, -a_j)$ $\quad I_n$ $\quad$ ~~$-F(-b_j) + rb_j$~~

$-F(b_j) + -rb_j \le -F(a_j) - ra_j$.

$-I_n \supset \overline{a_j b_j}\ E_{R,r}$

~~$G(x) = -F(-x) + rx$ | $r(b_i - a_i) \geq F(b_i) - F(a_i)$ ✓ on I_n~~

Start over: Try $G(x) = F(x) \cancel{+rx} - rx$.

Setting Sun lemma:

$G(x)$ on $[a,b]$

$$E = \{x \mid \exists h<0 : G(x+h) > G(x)\},$$

$\Rightarrow E$ open, $E = \bigcup (a_i, b_i)$

$$G(a_i) \geq G(b_i).$$

Apply Setting Sun to $G(x) = F(x) - rx \,|\, I_n$.

$$\left\{x : \exists h<0 : F(x+h) - r(x+h) > F(x) - rx,\ \frac{F(x+h) - F(x)}{h} < r\right\},$$

& $\bigcup (a_i, b_i)$

$$F(a_i) - ra_i = G(a_i) \geq G(b_i) = F(b_i) - rb_i$$

$$r(b_i - a_i) \geq F(b_i) - F(a_i)$$

On each (a_i, b_i) Apply Rising Sun

to $G(x) = F(x) - Rx \big|_{(a_i, b_i)}$.

then $\bigsqcup_k (a_{j,k}^{(n)}, b_{j,k}^{(n)}) = \{x \mid \exists h > 0 : F(x+h) - Rx - Rh > F(x) - Rx\}$

$$\frac{F(x+h) - F(x)}{h} > R$$

&

$$F(a_{j,k}) - Ra_{j,k} = G(a_{j,k}) \leq G(b_{j,k}) \leq F(b_{j,k}) - Rb_{j,k}.$$

$$R(b_{j,k} - a_{j,k}) \leq F(b_{j,k}) - F(a_{j,k}).$$

Let $E_{R,r} \cap I_n \subset \mathcal{O}_n = \bigsqcup_{j,k} (a_{j,k}, b_{j,k}) \subset I_n$.

$$m(\mathcal{O}_n) = \sum_{j,k} (b_{j,k} - a_{j,k})$$

$$\leq \frac{1}{R} \sum_{j,k} F(b_{j,k}) - F(a_{j,k})$$

$$\leq \frac{1}{R} \sum_{j} (F(b_j) - F(a_j)).$$

$$\leq \frac{r}{R} \sum_i (b_i - a_i)$$

$$m(O_n) \leq \frac{r}{R} m(I_n).$$

So $m(E_{R,r}) = \sum m(E_{R,r} \cap I_n) \leq \sum_n m(O_n)$

$$\leq \frac{r}{R} \sum_n m(I_n) = \frac{r}{R} m(O)$$

$$< \frac{r}{R} m(E_{R,r}) \frac{R}{r}$$