Given γ parametrized curve in $\mathbb{R}^2 = \mathbb{C}$ $\quad z : [a,b] \to \mathbb{R}^2$.

$z(t) = (x(t), y(t)) = x(t) + iy(t)$, continuous.

__Def:__ $L(\gamma) =$ length (if $z \in C'$, $L(\gamma) = \int_a^b \sqrt{\dot{x}^2 + \dot{y}^2}$)

$\rho \subset [a,b]$ partition

$a = t_0 < t_1 < \cdots < t_N = b$

$$L(\gamma) := \sup_{\rho} \sum_{\rho} |z(t_i) - z(t_{i-1})|$$

__Def:__ γ is rectifiable if $L(\gamma) < \infty$.

For general (not nec. continuous)

$F : [a,b] \to \mathbb{C}$, ρ $\quad \mathrm{Var}_{\rho} F = \sum_{\rho} |F(t_i) - F(t_{i-1})|$

$$\mathrm{Var}_{\mathcal{P}} F = \sum_{j=1}^{N} |F(t_j) - F(t_{j-1})|$$

<u>Def:</u> F is of <u>bdd variation</u> if

<u>total variation</u> $T_F := \sup_{\mathcal{P}} \mathrm{Var}_{\mathcal{P}} F < \infty$.

(If $F = \gamma$ ⟷ length → rectifiable.)

<u>Lemma:</u> Curve $\gamma : z(t) = (x(t), y(t))$ is rectifiable $\Leftrightarrow$ $x(t)$ & $y(t)$ bdd variation.

<u>pf:</u> $|a+ib| \le |a| + |b| \le 2|a+ib|$.

<u>Ex 1:</u> F increasing. For any $\mathcal{P}$,

$$\mathrm{Var}_{\mathcal{P}} F = \sum_{1}^{N} (F(t_j) - F(t_{j-1})) = F(b) - F(a).$$

<u>Ex 2:</u> If F' exists $\forall x \in (a,b)$ & F' bdd, then F is of <u>bdd variation</u>.

<u>pf:</u> Mean Value Thm $\Rightarrow$

$$|F(t_i) - F(t_{i-1})| \le \underbrace{|F'(c_i)|}_{M}\, |t_i - t_{i-1}|$$

$$\Rightarrow \mathrm{Var}_P F \le M \cdot (b-a)$$

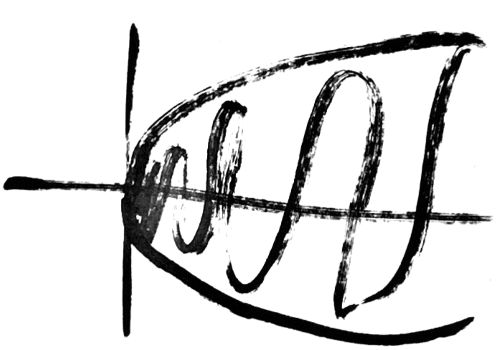
__Ex3:__ $F(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & 0 < x \le 1 \\ 0 & x = 0. \end{cases}$

← graph __connected__.
not path connected

Not of bdd variation, but F' exists!

(Building towards: F bdd variation $\Rightarrow F'$ exists a.e.)

__Ex4:__ $F(x) = \begin{cases} x^a \sin\frac{1}{x^b}, & 0 < x \le 1. \\ 0, & x = 0 \end{cases}$



For which $(a,b) \in \mathbb{R}^2_{>0}$ is F of bdd variation?

If $F : [a,b] \to \mathbb{R}$, let

positive variation ~~$P_F = \sup_{\rho} \sum |F(t_i) - F$~~

$$P_F = \sup_{\rho} \sum_{(+)} F(t_i) - F(t_{i-1})$$

& neg. variation: $N_F = \sup_{\rho} \sum_{(-)} \underbrace{-(F(t_i) - F(t_{i-1}))}_{\geq 0.}$

For $a \leq x \leq b$, write

$$P_F(a,x) = \sup_{\rho \subset [a,x]} \underbrace{\sum_{(+)} F(t_i) - F(t_{i-1})}_{P\,var_\rho(F)}.$$

Same $N_F(a,x)$, $T_F(a,x) \leftarrow$ total variation

<u>Lemma</u> (a la Jordan decomposition): if $F : [a,b] \to \mathbb{R}$ is of bdd variation, then $\forall\ a \leq x \leq b$ ① $F(x) - F(a) = P_F(a,x) - N_F(a,x)$

② $T_F(a,x) = P_F(a,x) + N_F(a,x)$

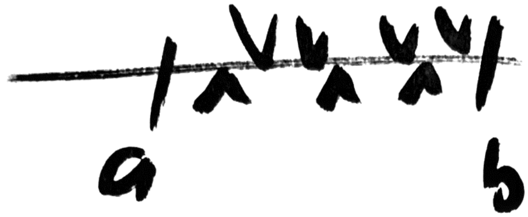
pf: since $P_F = \sup$, $\exists\ \rho \subset [a,x]$

s.t. $\left| \sum_{(+)} F(t_i) - F(t_{i-1}) - P_F(a,x) \right| < \varepsilon$

(9)

Same w/ $N_F(a,x)$, for $\mathcal{P}'$ other partition.

Take $\tilde{\mathcal{P}}$ = common refinement.



Obs: If $\tilde{\mathcal{P}}$ refinement of $\mathcal{P}$, then

$$PVar_{\tilde{\mathcal{P}}} F \geq PVar_{\mathcal{P}} F.$$

But for $\tilde{\mathcal{P}}$, $\sum_{(+)} F(t_i) - F(t_{i-1})$

$$\underline{F(x) - F(a)} = \_\_ - \sum_{(-)} -(F(t_i) - F(t_{i-1}))$$

For $T_F(a,x)$ +. | within 2ϵ of $P_F - N_F$.

<u>Thm</u>: $F : [a,b] \to \mathbb{R}$ is of bdd var

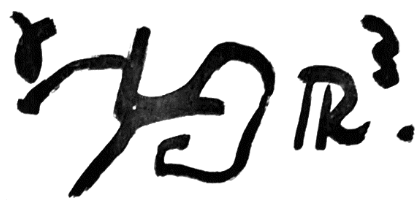
$\iff F = F_1 - F_2$ increasing, & bdd.

<u>pf ($\Leftarrow$)</u>: Obv   <u>pf ($\Rightarrow$)</u> $F_1(x) = PF(a,x) + F(a)$.

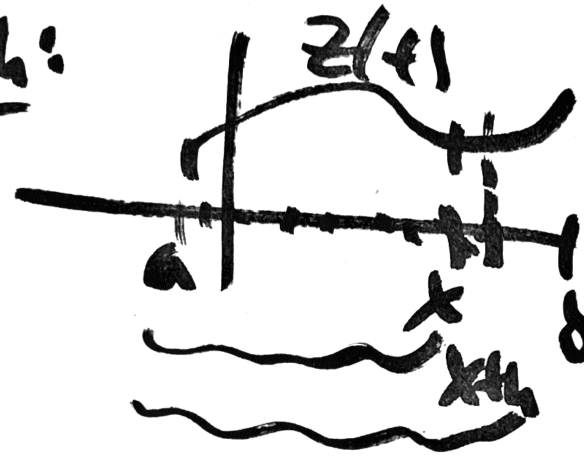
$F_2(x) = NF(a,x)$.

<u>Rmk</u>: If $F \in \mathbb{C}$ do Re / Im parts

Thm: Let $\gamma : [a,b] \to \mathbb{R}^3$ (or any metric space) be a cont curve of bdd variation. Then $T_F(a,x)$ is continuous.



Sketch:



Take partition of $[a, x+h]$

$$\left| Var_{F,P}(a,x+h) - T_F(a,x+h) \right| < \varepsilon$$

For h small enough,

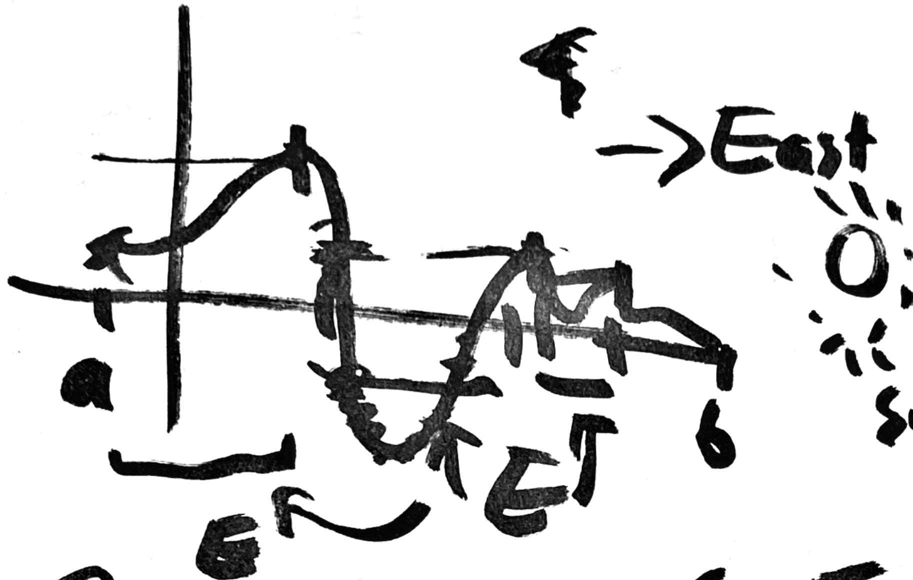
$$|Z(x) - Z(x+h)| < \varepsilon$$

If needed, refine partition to include x ....

Main Thm: If $F : [a,b] \to \mathbb{R}$ of bdd variation, then F' exists a.e.

Proof start with continuous functions (& can use increasing).

Lemma (Riesz): For $G:[a,b]\to\mathbb{R}$ continuous. Let $E = \{a<x<b \mid \exists h_x>0$ with ~~$G(x+h)>G(x)$~~ $G(x+h)>G(x)\}$.



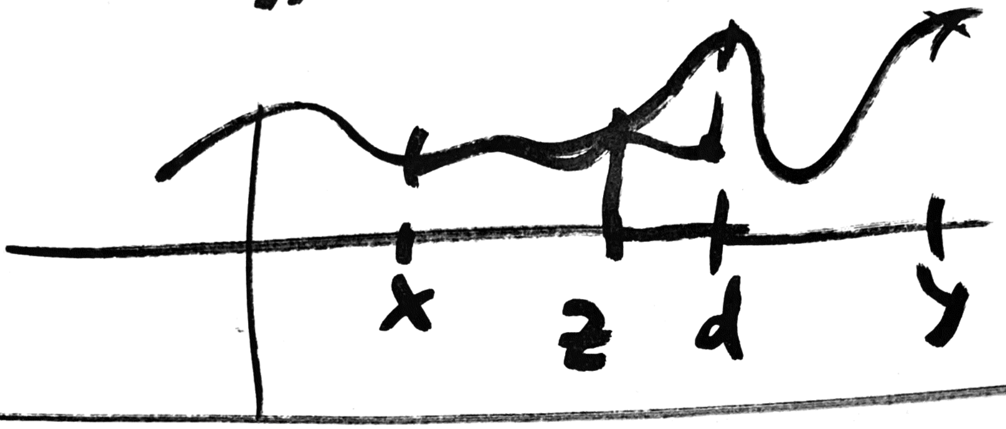
"sun rise set"

Then: If $E\neq\emptyset$, E is open. $\Rightarrow$ $E=\bigsqcup(a_j,b_j)$ & $G(b_j)=G(a_j)$. (except $a_j=a$).

Pf: If $x\in E$, $\exists\ x+h>x$ s.t. $G(x+h)>G(x)$. G cont $\Rightarrow$ (~~Inter Val Thm~~), $\exists\ [x,x+\varepsilon)\subset E$. extend ε as far as possible, $[x,d)\subset E$ & $d\notin E$. Claim: $G(d)>G(x)$.

$(\Rightarrow (x-\varepsilon,x)\subset E)$?

pf: If by contradiction $G(d) \le G(x)$. then $G|_{[x,d]}$ achieves max at $z \in [x,d) \subset E$. & $\exists\, y \in (z, b]$ with $G(y) > G(z) \ge G(d) \Rightarrow d \in E$.



Proved that E is open $\Rightarrow E = \sqcup (a_j, b_j)$

Claim: $x \in (a_j, b_j) \Rightarrow G(x) < G(b_j)$.

G cont $\Rightarrow G(a_j) \le G(b_j)$. $(x \to a_j^+)$

But $a_j \notin E$ (if $a_j \ne a$) $\nexists\, b_j$ with $G(a_j) < G(b_j)$. $\Rightarrow G(a_j) = G(b_j)$.

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