Recall: $\mathcal{H} = L^2(\rho)$ = Hilbert space

$|(f,g)| \le \|f\| \, \|g\|$ (Cauchy-Schwarz)

$\ell_{g \in L^2}(f) = (f,g) \in \mathbb{C}$ linear operator

Continuous $f_n \xrightarrow{\mathcal{H}} f \Rightarrow \ell(f_n) \to \ell(f)$.

bdd: $\|\ell\|_{\text{operator norm}} = \sup_{\substack{f \in \mathcal{H} \\ f \neq 0}} \frac{|\ell(f)|}{\|f\|}$ $<\infty$

$f \neq 0 \Leftrightarrow \|f\| \neq 0$.

Lemma: Cont $\Leftrightarrow$ bdd.

pf ($\Leftarrow$): $|\ell(f_n) - \ell(f)| \le |\ell(f_n - f)|$

$\le \underbrace{\|\ell\|}_{<\infty} \cdot \|f_n - f\| \to 0.$

pf ⇒): $f_n \to 0 \Rightarrow |\ell(f_n)| \to 0$, i.e. $\forall \varepsilon = 1\ \exists \delta$ s.t. $\|f_n\| < \delta \Rightarrow |\ell(f_n)| \leq 1$.

$\Rightarrow \|\ell\| \leq \frac{1}{\delta} < \infty$.

Riesz Rep: ℓ linear cont functional on $\mathcal{H} = L^2(\rho)$ Then $\exists!\, g \in \mathcal{H}$ s.t. $\ell(f) = (f, g)$, $\forall f \in \mathcal{H}$. ($\& \|\ell\| = \|g\|$).

Let $S = \{ f \in \mathcal{H} \mid \ell(f) = 0 \} \subset \mathcal{H}$.

Claim: S is closed. ($f_n \in S \to f \in \mathcal{H} \Rightarrow f \in S$).

pf: ℓ is cont.

Lemma: If $S \subset \mathcal{H}$ is closed, then $\mathcal{H} = S \oplus S^{\perp}$. i.e. $\oplus$ $\forall f \in \mathcal{H}$, $\exists!\, g \in S$ s.t. $f - g \in S^{\perp}$ & $f = g + (f - g)$.

$S^{\perp} = \{ g \in \mathcal{H} \mid (f,g) = 0, \forall f \in S \}$.

i.e. $\forall h \in \mathcal{H}$. ① $\exists! f_0 \in S$ s.t.

$$\inf_{f \in S} \|h - f\| = \|f_0 - h\|.$$

② $h - f_0 \in S^{\perp}$, i.e. $(h - f_0, f) = 0 \ \forall f \in S$.

pf ① Let $d = \inf_{f \in S} \|h - f\|$. If $d = 0$

$\Rightarrow h$ limit pt of $S \Rightarrow h \in S$ ✓.

Else $d > 0$, $\exists \{f_n\} \subset S$ $\|h - f_n\| \to d$.

Exercise (parallelogram law):

$$\|A + B\|^2 + \|A - B\|^2 = 2[\|A\|^2 + \|B\|^2].$$

$A = h - f_n$, $B = h - f_m$.

$$4\left\|h - \frac{f_n + f_m}{2}\right\|^2 + \|f_n - f_m\|^2 = 2[\|h - f_n\|^2 + \|h - f_m\|^2]$$

$\to 4d^2$

$$\|f_n - f_m\|^2 \le 2\left[\|h-f_n\|^2 + \|h-f_m\|^2\right] - 4d^2 \to 0.$$

$\Rightarrow f_n \to f_0 \in \mathcal{H}. \Rightarrow f_0 \in S.$

~~If $\|h - f_0\| = \|h - f_0'\|$, $f_0, f_0' \in S$~~

~~$\le \|h - f_0'\| + \|f_0 - f_0'\|$.~~

pf ②: let $u = h - f_0$, let $r \in \mathbb{R}$.

$f_0 - rf \in S$, $\|h - (f_0 - rf)\|^2 \ge \|h - f_0\|^2$

$= \|h-f_0\|^2 + r^2\|f\|^2 + 2\,\mathrm{Re}\, r(h-f_0, f)$.

$\Rightarrow r^2\|f\|^2 + 2r\,\mathrm{Re}(h-f_0, f) \ge 0.$

if $\mathrm{Re}(h-f_0,f) > 0$, $r \to 0^-$,

r^2 lower order than r $\Rightarrow$ ※.

$\Rightarrow \mathrm{Re} = 0$. Run again, $r \to ir$.

④ (Exercise)

Now if $f_0' \in S$ has $\|h-f_0\| = \|h-f_0'\|$.

$\Rightarrow (h-f_0, f_0') = 0$.

~~Pyth Th~~ $\Rightarrow \|h-f_0'\|^2 = \|h-f_0\|^2 + \|f_0-f_0'\|^2$.

Pythagorean Theorem

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pf (Reisz): If $S = \mathcal{H} \Rightarrow \ell = 0,\ g = 0$ ✓.

Else $\exists\, h \in S^\perp$, $h \neq 0$. Rescale $\|h\| = 1$.

Claim: $g = \overline{\ell(h)} \cdot h$.

pf: Look at $\ell(f) - (f, g)$.

$$= \ell(f) - \left(f, \overline{\ell(h)} \cdot h\right) = \ell(f) - \ell(h)(f,h).$$

$$= \ell(v) - \ell(h)(v,h).$$

$f = u + v$, $u \in S$, $v \in S^\perp$

Let $w = \ell(f)\cdot h - \ell(h)\cdot f$.

$w \in S$.

$\ell(w) = 0 \Rightarrow$

$$0 = (w,h) = \ell(f)(h,h) - \ell(h)(f,h)$$

$(h,h) = 1$

■

<u>Uniqueness:</u> $(f,g)=(f,g') \ \forall f$.

$\Rightarrow (f, g-g')=0 \quad \forall f = g-g'$

$\Rightarrow \|g-g'\|^2=0$.

<u>Op norm:</u>

$$\|\ell\| = \sup \frac{|\langle f,g\rangle|}{\|f\|} \overset{C\text{-}S}{\leq} \|g\|.$$

↑

$f=g \qquad (g,g)=\|g\|^2$

$\Rightarrow \|\ell\|=\|g\|$.