Quiz Nov 1, MIDTERM Nov 8,
Hwk due Tues's.

Recall: $(X_1, \mathcal{M}_1, \mu_1)$, $(X_2, \mathcal{M}_2, \mu_2)$.

$X = X_1 \times X_2$ (σ-finite), $\mathcal{M} = \langle \langle \mathcal{M}_1 \times \mathcal{M}_2 \rangle_A \rangle_\sigma$, $\mu = \mu_1 \times \mu_2$.

Prop: If $E \in \mathcal{M}$ Then (i) μ_2-a.e. x_2,

$E^{x_2} = \{x_1 \in X_1 \mid (x_1, x_2) \in E\} \in \mathcal{M}_1$

(ii) $f(x_2) = \begin{cases} \mu_1(E^{x_2}) & \text{if } \uparrow. \\ 0 & \text{otherwise.} \end{cases}$ is $\mathcal{M}_2$-meas.

(iii) $$\int_{X_2} f(x_2)\, d\mu_2(x_2) = \mu(E) = \int_{X_2} \int_{X_1} \chi_{E^{x_2}}(x_1)\, d\mu_1\, d\mu_2.$$

Thm (Fubini-Tonelli): If $f \in L^1(X \to \mathbb{C})$, then

(i) For μ_2-a.e. x_2, $f^{x_2}(x_1) = f(x_1, x_2)$ is $\mathcal{M}_1$-meas'ble

(ii) $\int_{X_1} f^{x_2}(x_1)\, d\mu_1$ is $\mathcal{M}_2$-measible & $\in L^1(X_2)$.

(iii) $\displaystyle\int_{X_2}\left[\int_{X_1} f(x_1,x_2)\,d\mu_1(x_1)\right]d\mu_2(x_2) = \int_X f\,d\mu.$

Pf: Assume $f \geq 0$. First $f = \chi_E$, $E \in \mathcal{M}$. (Prop). Finite linear $\Rightarrow$ $f =$ simple ≥ 0. MCT $\Rightarrow$ $f \geq 0$. $f = f^+ - f^-$. $f = \mathrm{Re} f + i\,\mathrm{Im} f$.

E.g.: $X_1 = \mathbb{R} \cap (0,\infty)$, $\mathcal{M}_1 \ni E \cap (0,\infty)$. μ_1

$X_2 = S^{d-1}$, Need $\underline{\mathcal{M}_2}$ & $\underline{\sigma}$ on X_2??.

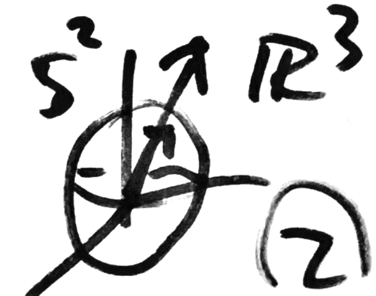
$X_1 \times X_2 = X = \mathbb{R}^d \setminus \{0\} \leftarrow \mathcal{M}$

$x \in \mathbb{R}^d \setminus \{0\}$, $|x| = r \in (0,\infty)$, $\dfrac{x}{|x|} = \gamma \in S^{d-1}$.

Thm: If $f \in L^1(\mathbb{R}^d)$, then

$$\int_{S^{d-1}}\int_0^\infty f(r\cdot\gamma)\,\underline{r^{d-1}\,dr}\,\underline{d\sigma(\gamma)} = \int_{\mathbb{R}^d} f(x)\,d\mu(x)$$

E.g. $d=3$,



$d\mu_1(r) \leftarrow \mu_1(E) = \int_E r^{d-1}\,dr.$

(2)

What should σ be? Given $E_2 \subset S^{d-1}$, let

$$\tilde{E}_2 = \{ r\gamma \mid 0<r<1,\ \gamma \in E_2 \}.$$

$\subset \mathbb{R}^d$. If $\tilde{E}_2 \in \mathfrak{M}$,

then we say $E_2 \in \mathfrak{M}_2$.

I.e. $\underline{\mathfrak{M}_2} = \{ E_2 \subset S^{d-1} : \tilde{E}_2 \in \mathfrak{M} \}$.

<u>Easy</u>: $\mathfrak{M}_2$ <u>is</u> a σ-alg.

Let $\sigma(E_2) = \underline{\mu_2(E_2)} = \underline{d}\ m(\tilde{E}_2)$.

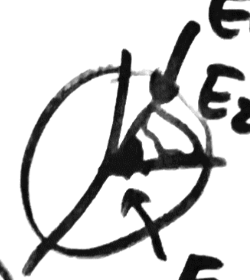
How does $r^{d-1}\,dr\,d\sigma \leftrightarrow dm$?

Let $E \subset \mathbb{R}^d$ be of the form $\{ r\gamma \mid r<1,\ \gamma \in E_2 \}$ $= (0,1) \times E_2$.

$$\underset{\tilde{E}_2}{\underset{\shortparallel}{m(E)}} \overset{?}{=} \int_{E_2} \underbrace{\int_0^1 r^{d-1}\,dr}_{\frac{1}{d}}\, d\sigma(\gamma) = \frac{1}{d} \cdot d \cdot m(\tilde{E}_2).$$

Now let $E = (0,b) \times E_2 \in \mathfrak{M}_1 \times \mathfrak{M}_2$.

$$\int_{E_2} \underbrace{\int_0^\delta \underline{r^{d-1}}\, dr}_{\frac{\delta^d}{d}} \, d\sigma \overset{?}{=} m(E).$$



$$\frac{\delta^d}{d}\,\sigma(E_2) = \frac{\delta^d}{\cancel{d}}\,\cancel{d}\, m(\tilde{E}_2)$$

$$\delta^d\, m(\tilde{E}_2) = m(E)$$

Check on other stuff...

<u>Cor</u>: If $f:\mathbb{R}^d \to \mathbb{C}$ is radial, & L^1 i.e. $f(x) = g(|x|)$ for $g:\mathbb{R}_+ \to \mathbb{C}$, then

$$\int_{\mathbb{R}^d} f\,dx = \int_{S^{d-1}} \int_0^\infty \underset{g}{f}(r\gamma)\, r^{d-1}\, dr\, d\sigma = \sigma(S^{d-1}) \cdot \int_0^\infty g(r)\, r^{d-1}\, dr.$$

<u>Cor</u>: For $a>0$, $\displaystyle\int_{\mathbb{R}^d} e^{-a|x|^2}\,dx = \cdots$ (Hwk).

<u>Cor</u>: $m(\mathbb{B}^d) = \dfrac{2\cdot\pi^{d/2}}{d\cdot\Gamma(d/2)}$.

Back to $f: \mathbb{R}^d \to \mathbb{R}$. for FTC.

Motivation: $F(x) = \int_a^x f(t)\,dt$ want:

$F'(x) = f(x)$,

$$\lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim \frac{1}{h} \int_x^{x+h} f.$$

$$= \underbrace{\lim_{\substack{I \ni x \\ |I| \to 0}}}\ \frac{1}{|I|} \int_I f \to f(x)$$

Def (Hardy-Littlewood Maximal Function):

Given $f \in L^1(\mathbb{R}^d)$, let

$$f^*(x) = \sup_{x \in B} \frac{1}{m(B)} \int_B |f| .$$

Thm: (i) f^* is meas'ble (ii) $f^*(x) < \infty$ a.e. x.

(iii) $m(x \mid f^*(x) > \alpha) \leq \dfrac{3^d}{\alpha} \|f\|_{L^1}$.

$\forall \alpha > 0$,

pf(i): Look at $\{x \mid f^*(x) > \alpha\} = E_\alpha$.

Claim: E_α is open.

pf: If $x \in E_\alpha \Rightarrow \exists B_x \ni x$ s.t.

$$\frac{1}{m(B_x)}\int_{B_x} |f| > \alpha. \text{ Also holds } \forall y \in B_x.$$

i.e. $f^*(y) > \alpha$. $\Rightarrow B_x \subset E_\alpha$.

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pf (i) $\Rightarrow$ (ii): $\{x \mid f^*(x) = \infty\} \subset \{x \mid f^*(x) > \alpha\}$

$$m(\{x \mid f^*(x) = \infty\}) \le m(\{x \mid f^*(x) > \alpha\}) \le \frac{3^d \|f\|}{\alpha} \to 0 \text{ as } \alpha \to \infty$$

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pf (iii): E_α is open. $\Rightarrow E_\alpha = \bigcup_n K_n$,

e.g. $K_n = \overline{B_n(0)} \cap \{x \mid d(x, E_\alpha^c) \ge \frac{1}{n}\}$.

Take K cpt $\subset E_\alpha$. $\forall x \in K \subset E_\alpha$,

$\exists B_x$ s.t. $\frac{1}{m(B_x)}\int_{B_x} |f| > \alpha$. So $K \subset \bigcup_x B_x$

$\boxed{m(B_n) \le \frac{1}{\alpha}\int_{B_n} |f|} \longleftrightarrow$ (*)

$\Rightarrow \exists B_1, \dots, B_N$ with $K \subset \bigcup_{n=1}^{N} B_n$.

$$m(K) \le \sum_{n=1}^{N} m(B_n) < \frac{N}{\alpha} \|f\|_{L^1} \longleftarrow \text{No good!}$$

Lemma: Given $\mathcal{B} = \{B_1, \ldots, B_N\}$ balls, $\exists$ subset $S \subset \mathcal{B}$ of disjoint balls s.t.

$$m\Big(\bigcup_{B \in \mathcal{B}} B\Big) \leq \underline{3}^d \cdot \sum_{B \in S} m(B)$$

pf: If $B_{r_1}(x_1) \cap B_{r_2}(x_2) \neq \emptyset$ & $r_1 \geq r_2$

$\Rightarrow B_{3r_1}(x_1) \supset B_{r_2}(x_2)$

So let B_1 = largest radius ball in $\mathcal{B}$. Put it in S, throw out from $\mathcal{B}$ anything intersecting B_1. Let B_2 = largest remaining. Continue . . . .