Review: $\varphi = \sum_{k=1}^{N} c_k \chi_{E_k}$ canonical form, $\varphi : X \to \mathbb{R}_{\geq 0}$, $(X, \mathcal{M}, \mu)$.

$$\Rightarrow \int \varphi \, d\mu = \sum_{k=1} c_k \, \mu(E_k).$$

Proved: indep of rep'n.

Def: $f : X \to \overline{\mathbb{R}}_{\geq 0}$ is measable $\Rightarrow \int f \, d\mu = \sup_{0 \leq \varphi \leq f} \int \varphi \, d\mu$.

Monotone Conv Thm: $f_n \geq 0$, $f_n \leq f_{n+1}$, $f_n \to f$ a.e. $\Rightarrow$ $\int f_n \to \int f$.

Corollaries:
- $f \geq 0$: $\int f = 0 \iff f = 0$ a.e.
- $f_n \geq 0$: $\int \sum_n f_n = \sum_n \int f_n$.

Fatou's Lemma: $f_n \geq 0$: $\liminf \int f_n \geq \int \liminf f_n$.

$f_1 : 1, 2 \to \int f_1 = 3$
$f_2 : 0, 3$ $\quad \int f_2 = 3$.
$\inf f : 0, 2.$ $\quad \int \inf f = 2$

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$$\int \liminf f_n \leq \liminf \int f_n.$$

Reverse Fatou: $f_n \geq 0$ & $\exists g$ with $\int g < \infty$ s.t. $f_n \leq g$. Then $\limsup \int f_n \leq \int \limsup f_n$. pf: $g - f_n \geq 0 \Rightarrow \int \liminf (g - f_n) \leq \liminf \int (g - f_n)$

$g - \limsup f_n \leq \quad \int g - \int f_n$

$f : X \to \overline{\mathbb{R}}$, $f^+ = \max(f, 0)$, $f^- = \max(-f, 0)$.

$f = f^+ - f^-$. If $\int f^+ < \infty$ & $\int f^- < \infty \Rightarrow \int f = \int f^+ - \int f^-$.

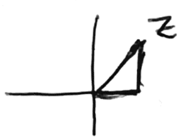
$\int |f| < \infty$, $\quad |f| = f^+ + f^-$.

$f : X \to \mathbb{C}$: $f = \mathrm{Re} f + i \, \mathrm{Im} f$.

If $\int |f| < \infty \Rightarrow \int f = \int \mathrm{Re} f + i \int \mathrm{Im} f$.

$\Updownarrow$

$\int |\mathrm{Re} f| < \infty$ & $\int |\mathrm{Im} f| < \infty$.



$|z| \leq |\mathrm{Re}(z)| + |\mathrm{Im}(z)| \leq 2|z|$ (1)

"**Def:**" $L^1(X,\mu) = \{ f: X \to \mathbb{C} \mid f \text{ integrable}, \int |f| < \infty \}$.

Is a vector space$/\mathbb{C}$. $0 \in L^1$, $f,g \in L^1 \Rightarrow f+g \in L^1 \Rightarrow cf \in L^1$.

Thm: "f has small tails". $f \in L^1$ & X σ-finite Then $\forall \varepsilon > 0$, $\exists E \in \mathcal{M}$ with $\mu(E) < \infty$ s.t. $\int_{E^c} |f| < \varepsilon$.

(E.g. $X = \mathbb{R}^d$, $E = B(0,N)$.)

pf: Assume $f \geq 0$. Let $X = \bigcup_n E_n$, $\mu(E_n) < \infty$. Then $E_n \subset E_{n+1}$

$f_n = f \cdot \chi_{E_n} \Rightarrow f_n \nearrow f$. MCT$\Rightarrow$ $\underset{\overset{\shortparallel}{\int_{E_n} f}}{\int f_n} \to \int f < \infty$.

So $\forall \varepsilon > 0$ $\exists n$ s.t. $\underbrace{\int f - f_n}_{= \int_{E_n^c} f} < \varepsilon$. Take $E = E_n$. ∎

Thm: $f \in L^1 \Rightarrow \forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $\forall E \in \mathcal{M}$ with $\mu(E) < \delta$, $\int_E |f| < \varepsilon$. (Absolute continuity of integral).

pf: Take $f \geq 0$. Let $E_N = \{ x \mid f(x) \leq N \}$. Let $f_N = f \cdot \chi_{E_N} \nearrow f$ a.e.

MCT $\Rightarrow$ $\int f \chi_{E_N} \to \int f$ so $\exists N$ s.t. $\int (f - f_N) < \frac{\varepsilon}{2}$. Pick $\delta < \frac{\varepsilon}{2N}$. Assume $E \in \mathcal{M}$ has $\mu(E) < \delta$. Look at

$$\int_E f = \int_E (f - f_N) + \int_E f_N \leq \underbrace{\int (f - f_N)}_{\frac{\varepsilon}{2}} + \underbrace{N \cdot \mu(E)}_{N \cdot \frac{\varepsilon}{2N}} < \varepsilon.$$

Dominated Convergence Thm: If $\{f_n\}$ meas, $f_n \to f$ a.e. & $\exists\, g \in L^1$ s.t. $\forall n$, $|f_n| \le g$ a.e. Then: $f \in L^1$ & $f_n \xrightarrow{L_1} f$, i.e. $\int |f - f_n| \to 0$. (& hence $\int f_n \to \int f$).

pf: limsup f_n & liminf f_n are meas'ble $\Rightarrow$ f is meas'ble.
Each $|f_n| \le g$, $\Rightarrow |f| \le g$. $\Rightarrow \int |f| \le \int g \Rightarrow f \in L^1$.
$|f_n| \le g \Rightarrow |f - f_n| \le |f| + |f_n| \le 2g$. Reverse Fatou $\Rightarrow$
$0 \le \limsup \int |f - f_n| \le \int \limsup |f - f_n| = 0$.

Thm: (Reisz-Fischer): $L^1(X, \mu)$ is a complete metric space

L^1 norm $\|f\| = \int |f|$, $\|f+g\| \le \|f\| + \|g\|$, $\|f\| \ge 0$.
$\|f\| = 0 \Rightarrow f = 0$ a.e. $\leftarrow$ Need!!!

Def: $\mathcal{L}^1(X, \mu) = \{ f : X \to \mathbb{C} \mid \int |f| < \infty \} / \sim$
where $f \sim g \Leftrightarrow f - g = 0$ a.e.

$f \in L^1$
$[f] \in L^1$.

"$\mathcal{L}^1$ functions do not have values".

$\int f = \int g \not\Rightarrow f = g$

L^1 metric: $d(f, g) = \|f - g\|$ is a metric.

So Reisz-Fischer: If $\{f_n\} \subset L^1$ is Cauchy, i.e., $\forall \varepsilon > 0\ \exists N\ \forall n, m \ge N$, $\|f_n - f_m\| < \varepsilon$. Then $\exists f \in L^1$ s.t. $f_n \xrightarrow{L_1} f$ i.e. $\|f_n - f\| \to 0$. ($\not\Rightarrow f_n \to f$ a.e.).

Counterexample to a.e. convergence (under L^1 convergence).

$f_1 = \chi_{[0,1]}$, $f_2 = \chi_{[0,\frac{1}{2}]}$, $f_3 = \chi_{[\frac{1}{2},1]}$, $f_4 = \chi_{[0,\frac{1}{4})}$,

$f_5 = \chi_{[\frac{1}{4},\frac{1}{2}]}$, $f_6 = \chi_{[\frac{1}{2},\frac{3}{4})}$, $f_7 = \chi_{[\frac{3}{4},1]}$, ...

$\forall x \in [0,1]$, $f_n(x) = 0$ & 1 i.o. No limiting function (a.e.)

But in L^1, $f_n \to 0$. Because $\int |f_n| \to 0$.

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pf. Know $\forall \varepsilon > 0\ \exists N\ \forall n,m \geq N$, $\|f_n - f_m\| < \varepsilon$. Take $\varepsilon = \frac{1}{2^k}$.

$\exists N = n_k$ s.t. $\forall n,m \geq n_k$, $\|f_n - f_m\| < \frac{1}{2^k}$. $\{f_{n_k}\}$ is a subseq for which $\|f_{n_k} - f_{n_{k+1}}\| < \frac{1}{2^k}$.

Let
$$f(x) = f_{n_1}(x) + \sum_{k=1}^{\infty} \left(f_{n_{k+1}}(x) - f_{n_k}(x) \right).$$

Look at
$$F_K(x) = f_{n_1}(x) + \sum_{k=1}^{K-1} \left(f_{n_{k+1}}(x) - f_{n_k}(x) \right) = f_{n_K}(x).$$

Each
$$|F_K(x)| \leq G_K(x) = |f_{n_1}(x)| + \sum_{k=1}^{K-1} \left| f_{n_{k+1}}(x) - f_{n_k}(x) \right|$$

$$G_K \nearrow G(x) = |\ \cdot\ | + \sum^{\infty} |\ \cdots\ |$$

But $G \in L^1$ since
$$\int G = \|f_{n_1}\| + \sum_{k=1}^{\infty} \underbrace{\|f_{n_{k+1}} - f_{n_k}\|}_{< \frac{1}{2^k}} < \infty.$$

$\Rightarrow G < \infty$ a.e. $\Rightarrow$ series defining f converges abs. a.e.

$\underbrace{f_{n_K} \to f}$ a.e. Note:
$$\underbrace{\left| f(x) - f_{n_K}(x) \right|}_{(4)} = \left| \sum_{k=K}^{\infty} f_{n_{k+1}}(x) - f_{n_k}(x) \right|$$

$\to 0$ a.e.

$$\leq \sum_{k=K}^{\infty} \left| f_{n_{k+1}}(x) - f_{n_k}(x) \right| \leq \underbrace{G(x)}.$$

Dominated Convergence $\Rightarrow$ $\|f - f_{n_k}\| \to 0$.

Easy finish: If $\varepsilon > 0$, let N be s.t. $\forall n, m \geq N$, $\|f_n - f_m\| < \frac{\varepsilon}{2}$, Choose k large enough that $\|f - f_{n_k}\| < \frac{\varepsilon}{2}$, & $n_k > N$. Then $\|f_n - f\| \leq \underbrace{\|f_n - f_{n_k}\|}_{<\varepsilon/2} + \underbrace{\|f - f_{n_k}\|}_{\varepsilon/2} < \varepsilon$.

Cor: $\{f_n\}$ seq, $f_n \to f$ in $\mathcal{L}^1$. Not true that $f_n \to f$ a.e. But $\exists \{f_{n_k}\} \subset \{f_n\}$ s.t. $f_{n_k} \to f$ a.e.
