Recall: (X,d) metric space, $d(x,y)\ge 0$ & $d: X\times X\to\mathbb{R}$

$d(x,y)=0 \Rightarrow x=y$, $d(x,y)=d(y,x)$, $d(x,z)\le d(x,y)+d(y,z)$.

- Induces a topology, $B_r(x)=\{y\in X \mid d(x,y)<r\}$.
- E bdd if: $\exists R<\infty$ & $x\in X$ with $E\subset B_R(x)$.

  OR • $\operatorname{diam}(E)=\sup\{d(x,y)\mid x,y\in E\}<\infty$.

Thm (Heine-Borel): Let $E\subset X$. TFAE.

① E is complete, i.e. every Cauchy sequence $\{x_n\}\subset E$, $x_n\to x\in E$.
& totally bdd, i.e. $\forall\varepsilon>0$, $\exists B_\varepsilon(x_1),\dots,B_\varepsilon(x_N)$ with
$$E\subset\bigcup_{j=1}^{N} B_\varepsilon(x_j).$$

② "Bolzano-Weierstrass": every $\{x_n\}\subset E$ has a convergent (in E) subsequence

③ E is compact: Any open cover has finite subcover.

pf: ① ⇒ ② Let $\{x_n\}\subset E$ be infinite sequence. $\exists$ Cover $E\subset\bigcup_{j=1} B_{1/2}(y_j^{(1)})$ some $B_{1/2}(y^{(1)}_{J_1})$ contains ∞ly many $\{x_n\}$'s.

Cover $E\cap B_{1/2}(y^{(1)}_{J_1})\subset\bigcup_{j=1}^{N_2} B_{1/4}(y^{(2)}_j)$ " " " " " " "

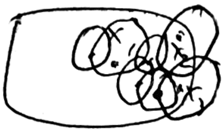
$B_{1/2}(y^{(1)}_{J_1})\supset B_{1/4}(y^{(2)}_{J_2})\supset\cdots\supset B_{1/2^k}(y^{(k)}_{J_k})$. For each k, let n_k be least n s.t. $x_{n_k}\in B_{1/2^k}(y^{(k)}_{J_k})$, $n>n_{k-1}$.

This is a Cauchy seq. $\Rightarrow x_{n_k}\to x\in E$

pf ② ⇒ ①: ¬① ⇒ ¬②: If $\exists \{x_n\} \subset E$ Cauchy with no limit in E, then there is no convergent subsequence.

If E not totally bdd, $\exists\, \varepsilon > 0$ s.t. $\nexists$ finite ε-ball cover of E. Take $x_1 \in E$, take any $x_n \in E \setminus \bigcup_{j=1}^{n-1} B_\varepsilon(x_j)$.

All x_n have distance $> \varepsilon$ from each other. $d(x_n, x_m) \geq \varepsilon$



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pf ③ ⇒ ②: Let $\{x_n\} \subset E$. Cover $E \subset \bigcup_{j=1}^{N} B_{1/2}(y_j)$

Continue as before. $\square$

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pf ② ⇒ ③: Assume $E \subset \bigcup_{\alpha \in A} \mathcal{O}_\alpha$. Claim $\exists\, \varepsilon > 0$ s.t. $\forall x \in E$, $B_\varepsilon(x) \cap E \neq \emptyset \Rightarrow B_\varepsilon(x) \subset \mathcal{O}_\alpha$ for some $\alpha \in A$.

If not, $\forall\, n \geq 1$, $\exists\, B_n = B_{1/2^n}(x_n) \cap E \neq \emptyset$ & B_n not contained in any $\mathcal{O}_\alpha$. But $\exists$ subseq $\{x_{n_j}\}$ with $x_{n_j} \to x \in E$.

But $x \in E$ so $x \in$ some $\mathcal{O}_\alpha$. $\Rightarrow \exists\, B_\varepsilon(x) \subset \mathcal{O}_\alpha$ some $\mathcal{O}_\alpha$.

$\Rightarrow$ some $\frac{1}{2^N}$-Ball are contained in $\mathcal{O}_\alpha$.

$\Rightarrow E$ can be covered by finitely many ε-balls, each in some $\mathcal{O}_\alpha$. $\Rightarrow$ finite subcover.

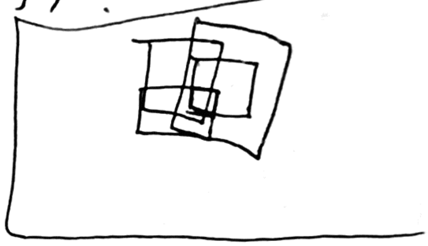
Def: $(X, \mathcal{A}, \mu_0)$ is a pre-measure space: if $\mathcal{A}$ is algebra ~~(σ-finite)~~ (closed under complement & finite unions & intersections). & μ_0 = premeasure $\Rightarrow$ $\mu_0 : \mathcal{A} \to \mathbb{R}_{\geq 0}$ & countably additive. i.e. $\emptyset \in \mathcal{A}$.

If $E_j \in \mathcal{A}$ & $\bigsqcup_{j=1}^{\infty} E_j \in \mathcal{A} \Rightarrow \mu_0\left(\bigsqcup E_j\right) = \sum \mu_0(E_j)$.

Thm: $(X, \mathcal{A}, \mu_0)$ pre-measure $\Rightarrow$ (X, μ_*) is exterior meas. space.

where:

$$\mu_*(E) = \inf_{\substack{E \subset \bigcup_{j=1}^{\infty} E_j \\ E_j \in \mathcal{A}}} \sum \mu_0(E_j).$$



Moreover, $\forall E \in \mathcal{A}$, $\mu_*(E) = \mu_0(E)$.

Pf: $\mu_*(\emptyset) = 0$ ✓. If $E_1 \subset E_2$, $\mathcal{A}$-cover of $E_2 \Rightarrow$ one of E_1

$\Rightarrow \mu_*(E_1) \leq \mu_*(E_2)$ ✓. Countable subadditivity:

If $\bigcup E_j = E$, assume $\mu_*(E_j) < \infty$. Each $E_j \subset \bigcup_k A_k^{(j)}$, $A_k^{(j)} \in \mathcal{A}$

s.t. $\sum_k \mu_0(A_k^{(j)}) \leq \mu_*(E_j) + \frac{\varepsilon}{2^j}$. So $E \subset \bigcup_j \bigcup_k A_k^{(j)}$.

$$\mu_*(E) \leq \sum_j \sum_k \mu_0(A_k^{(j)}) \leq \sum_j \mu_*(E_j) + \varepsilon.$$

If $E \in \mathcal{A}$, $\mu_*(E) \leq \mu_0(E)$ ✓. Let $E \subset \bigcup A_j$, $A_j \in \mathcal{A}$, set ~~$E_k = E \cap (A_k \cup$~~

$E_k = E \cap \left(A_k \setminus \bigcup_{j=1}^{k-1} A_j\right) \in \mathcal{A}$. $\mu_0(E) \leq \mu_0\left(\bigsqcup E_k\right) = \sum \mu_0(E_k) \leq \sum \mu_0(A_k)$

True for any $\mathcal{A}$-cover (3) $\Rightarrow \mu_*(E) \geq \mu_0(E)$.

Recall: in $\mathbb{R}^d$, the finish was to say: $E \subset \mathbb{R}^d$ is ~~open~~ measurable if: $\forall \varepsilon > 0 \; \exists \, \mathcal{O} \supset E$ with $\mu_*(\mathcal{O} \setminus E) < \varepsilon$.

Def: Given exterior measure space (X, μ_*), a set $E \subset X$ is Caratheodory measurable if:

$$\forall A \subset X, \quad \mu_*(A) = \mu_*(A \cap E) + \mu_*(A \cap E^c).$$

Lemma: If $(X, \mathcal{A}, \mu_0)$ pre-measure & μ_* extension then every $E \in \mathcal{A}$ is measurable.

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pf: let $E \in \mathcal{A}$ & $A \subset X$ arbitrary. $\mu_*(A) \leq \mu_*(A \cap E) + \mu_*(A \cap E^c)$ ✓

Cover A by $E_j \in \mathcal{A}$ s.t. $\sum_j \mu_0(E_j) \leq \mu_*(A) + \varepsilon$.

$$\mu_*(A \cap E) + \mu_*(A \cap E^c) \leq \sum_j \left(\mu_0(E_j \cap E) + \mu_0(E_j \cap E^c) \right)$$

$\bigcup_j$ covers $A \cap E$ covers $A \cap E^c$.

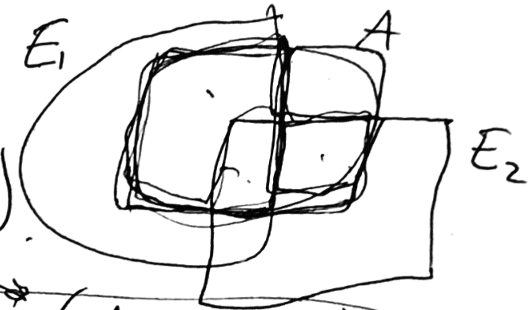
Thm (Caratheodory): If (X, μ_*) is an exter measure, then ① $\mathcal{M} = \{ E \subset X \mid \text{measurable} \}$ is a σ-alg. &

② $\mu = \mu_* |_{\mathcal{M}}$ is a measure. (countably additive)

pf ①: If $E \in \mathcal{M} \Rightarrow E^c \in \mathcal{M}$ because def is symmetric.

Claim: $E_1, E_2 \in \mathcal{M} \Rightarrow E_1 \cup E_2 \in \mathcal{M}$.

Let $A \subset X$ arbitrary. $E_1 \in \mathfrak{M}$.



$$\mu_*(A) = \underline{\mu_*(A\cap E_1)} + \mu_*(A\cap E_1^c).$$

$E_2 \in \mathfrak{M}$

$$= \mu_*(A\cap E_1\cap E_2) + \mu_*(A\cap E_1\cap E_2^c) + \mu_*(A\cap E_1^c\cap E_2) + \mu_*(\underbrace{A\cap E_1^c\cap E_2^c}_{})$$

$\mu_*(A\cap(E_1\cup E_2)^c)$

$$\geq \mu_*(A\cap(E_1\cup E_2)) + \mu_*(A\cap(E_1\cup E_2)^c).$$

& $\mu_*(A) \leq \mu_*(A\cap(E_1\cup E_2)) + \mu_*(A\cap(E_1\cup E_2)^c)$