Recall: $f : \mathbb{R}^d \to \mathbb{R}$ measurable if:

$f^{-1}(\mathcal{O}_{\text{open}}) \in \mathcal{M}$.

Def: Characteristic function $f = \chi_{E \in \mathcal{M}} = \begin{cases} 1 & x \in E \\ 0 & \text{oth.} \end{cases}$

Simple function $f = \sum_{j=1}^{N} a_j \chi_{E_j}$.

Thm: $\sup f_n$ is measurable if f_n's are.

Lemma: If f & g are measurable $f+g$ measurable.

pf: $(f+g)^{-1}((a,\infty)) = \{f+g > a\} = \bigcup_{r \in \mathbb{Q}} \left(\{f > a-r\} \cap \underbrace{\{g > r\}}_{} \right)$

$f+g > a \Rightarrow \underbrace{f > a+}_{}$

Countable union of $\in \mathcal{M}$.

Lemma: If f measurable, f^2 is too. $\boxed{f \circ \Phi \times \quad \Phi \circ f \checkmark}$

pf: Use $f^2 = (\cdot^2) \circ f$; $\underline{\{f^2 > a\} = \{f > \sqrt{a}\} \cup \{f < -\sqrt{a}\}}$.

If Φ cont, then $\Phi^{-1}(\mathcal{O}) =$ open.

But $\Phi^{-1}(E \in \mathcal{M})$ need not be $\in \mathcal{M}$.

Measurable: $f^{-1}(\mathcal{O}) \in \mathcal{M}$.

Lemma: If f & g meas'ble & finite valued. Then $f \cdot g$ meas'ble.

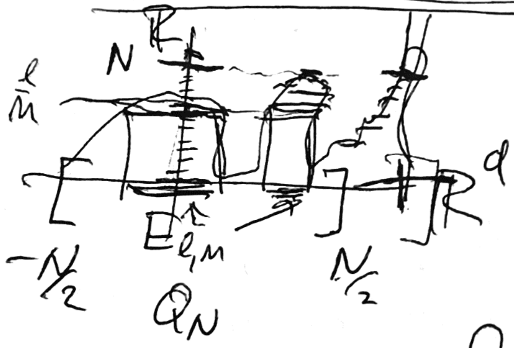
pf: $\frac{1}{4}\left((f+g)^2 - (f-g)^2\right) = fg$.

Def: $f = g$ a.e. if $m(\{x \in E \mid f(x) \neq g(x)\}) = 0$.

Rmk: pfs all same for a.e. x instead of all x.

Structure of meas'ble functions:

Thm: If $f \geq 0$ on E, meas'ble $\Rightarrow \exists \{\varphi_k\}$ of simple functions, $\varphi_k(x) \to f(x)$ pointwise, $\varphi_k \geq 0$, $\varphi_k(x) \leq \varphi_{k+1}(x)$.



Let Q_N = cube of side length N on O.

$$F_N(x) = \begin{cases} f(x) & x \in Q_N \ \& \ f(x) < N \\ N & x \in Q_N \ \& \ f(x) \geq N \\ 0 & x \notin Q_N \end{cases}$$

Let $E_{\ell,M} = \left\{ x \in E \cap Q_N \;\middle|\; \frac{\ell}{M} \leq F_N(x) < \frac{\ell+1}{M} \right\}$, $0 \leq \ell \leq NM$

Let $F_{N,M}(x) = \sum_{0 \leq \ell < NM} \frac{\ell}{M} \cdot \chi_{E_{\ell,M}} + \cancel{N \cdot \chi_{\{x \in Q_N \mid F_N(x) = N\}}}$

$$\varphi_k(x) = F_{2^k, 2^k}(x)$$

Thm: Let f be meas'ble : $E \subset \mathbb{R}^d \to \mathbb{R}$. Then $\exists$ seq φ_k simple functions, $\varphi_k(x) \to f(x)$ $\forall x \in E$.
& $|\varphi_k(x)| \le |\varphi_{k+1}(x)|$. (Obs: $|\varphi_k(x)| \le |f(x)|$).

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pf: $f = f^+ - f^-$, $f^+(x) = \max(f(x), 0) \ge 0$
$f^-(x) = +\max(-f(x), 0) \ge 0$.

$\exists \varphi_k^+ \to f^+$, $\exists \varphi_k^- \to f^-$. Let $\varphi_k = \varphi_k^+ - \varphi_k^-$.

$|\varphi_k(x)| = \varphi_k^+ + \varphi_k^-$

$\boxed{\mathcal{M}(\mathbb{R}^d) = \text{meas'ble functs} \quad C(\mathbb{R}^d), \ \mathcal{M}}$

Def: A step function $f = \sum_{k=1}^{N} a_k \chi_{R_k}(x)$

finite sum of rectangles

Thm: If $f \in \mathcal{M}$, then $\exists \{\psi_k\}$ step functions s.t.
$\psi_k(x) \to f(x)$ a.e. $x \in \mathbb{R}^d$.

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pf: Assume $f = \chi_E$. Recall given $E \in \mathcal{M}$, $\forall \varepsilon > 0$ $\exists$
$Q_1, \dots, Q_N$ cubes s.t. $m(E \triangle \bigcup_{j=1}^{N} Q_j) < \varepsilon$.

Q_j

$\bigcup_{j=1}^{N} Q_j = \bigsqcup_{j=1}^{M} \tilde{R}_j$, Take very slightly smaller rectangles $R_j \subset \tilde{R}_j$

s.t. $m(E \triangle \bigsqcup_{j=1}^{M} R_j) < \varepsilon$. Do this with $\varepsilon = \frac{1}{2^k}$.

Set $\psi_k(x) = \sum_{j=1}^{M_k} \chi_{R_j}(x)$

let $E_k = \{ x \in \mathbb{R}^d \mid \psi_k(x) \neq f = \chi_E \} = E \triangle \bigcup_{j=1}^{M_k} R_j$.

$m(E_k) < \frac{1}{2^k}$. let $F_K = \bigcup_{k > K} E_k$

$m(F_K) \leq \frac{1}{2^K}$. let $F = \bigcap_{K=1}^{\infty} F_K$

$\{ x \mid \underline{\psi_k(x)} \not\to \chi_E \} \subset F$. & $\underset{\substack{\| \\ 0}}{m(F)} \leq m(F_K) \leq \frac{1}{2^K}$.

For arbitrary f, $\exists$ φ_k simple functions $\varphi_k \to f$

Each $\varphi_k = \lim_{\ell \to \infty} \psi_\ell^{(k)}$, $\psi_{M_k}^{(k)} \to f$ a.e.

Given $\varepsilon > 0$. $\exists N : \forall k > N$, $|\varphi_k(x) - f(x)| < \varepsilon/2$.

$\exists M_k : \forall \ell > M_k$ $|\varphi_k(x) - \underline{\psi_\ell^{(k)}(x)}| < \varepsilon/2$.

φ $\psi_{M_k}^{(k)}(x)$

φ