Recall: Def: $m_*(E) = \inf_{E\subset\bigcup_{j=1}^{\infty} Q_j \text{ closed cubes}} \sum_{j=1}^{\infty} |Q_j|$ (exterior meas).

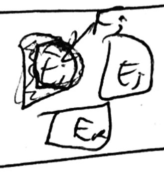
Def: $E\subset\mathbb{R}^d$ is measurable iff $\forall\varepsilon>0$ $\exists\,\mathcal{O}$ open $\supset E$ s.t. $m_*(\mathcal{O}\setminus E)<\varepsilon$. $\mathcal{M}$ = set of meas'ble sets.

Prop: $\mathcal{M}$ is a σ-algebra, i.e., closed under count. unions & complements

Borel σ-alg: $\langle\mathcal{O}\rangle$. Lebesgue + meas 0 sets.

Thm: If $E_j\in\mathcal{M}$, pairwise disjoint, & $E=\bigsqcup_{j=1}^{\infty}E_j$ Then

$$m(E)=\sum_{j=1}^{\infty}m(E_j).$$



pf: $(\le)$ done, need $(\ge)$. First assume E_j bdd. $E_j^c\in\mathcal{M}$. $\Rightarrow\exists\,\mathcal{O}_j\supset E_j^c$ open with $\underset{\overset{\|}{m(E_j\setminus F_j)}}{m}(\mathcal{O}_j\setminus E_j^c)<\frac{\varepsilon}{2^j}$. let $F_j=\mathcal{O}_j^c$ closed $\subset E_j$.

Each $F_j\cap F_k=\emptyset$. $\Rightarrow d(F_j,F_k)>0\Rightarrow m(F_1\sqcup\cdots\sqcup F_N)$

$m(E)\ge m(F_1\sqcup\cdots\sqcup F_N)=\sum_{j=1}^{N}m(F_j)$

$m(E_j)\le\underline{m(E_j\setminus F_j)}+m(F_j)$

$\ge\sum_{j=1}^{N}\left(m(E_j)-\frac{\varepsilon}{2^j}\right)$

$\Rightarrow m(E)\ge\sum_{j=1}^{N}m(E_j)-\varepsilon$. $N\to\infty$, $\varepsilon\to0$.

For arbitrary E_j, let $S_1=\overline{B_1(0)}$, $S_k=\overline{B_k(0)}\setminus\overline{B_{k-1}(0)}$

$E_j=\bigsqcup_{k=1}^{\infty}\underbrace{E_j\cap S_k}_{\text{bdd }\in\mathcal{M}}$ & $E=\bigsqcup_{j=1}^{\infty}\bigsqcup_{k=1}^{\infty}E_j\cap S_k$ use bdd case ∎.

Def. If $E_1 \subset E_2 \subset \cdots$ $E = \bigcup E_k$, $E_k \nearrow E$.

If $E_1 \supset E_2 \supset \cdots$ $E = \bigcap E_k$ $E_k \searrow E$.

Cor: If $E_j \in \mathcal{M}$, ① If $E_k \nearrow E$, then $m(E_k) \to m(E)$.

② If $E_k \searrow E$, then (& some $m(E_k) < \infty$), $m(E_k) \to m(E)$.

(For ②, look at $E_n = (n, \infty)$. $\bigcap E_n = E = \emptyset$.)

pf: ① Let $F_1 = E_1$, $F_k = E_k \setminus E_{k-1}$.

E.

Then $E = \bigsqcup_{k=1}^{\infty} F_k$, $m(E) = \lim_{N\to\infty} \underbrace{\sum_{k=1}^{N} m(F_k)}_{m(E_N)} = \lim_{N\to\infty} m\left(\bigsqcup_{k=1}^{N} F_k\right)$

② Assume $m(E_1) < \infty$.

E_1, let $F_1 = E_1 \setminus E_2$, $F_2 = E_2 \setminus E_3$ ...

Then $E_1 = \bigsqcup_{j=1}^{\infty} F_j \sqcup E \Rightarrow m(E_1) = \lim_{N\to\infty} \underbrace{\sum_{j=1}^{N-1} m(F_j)}_{m(E_1) \cancel{- m(E_2) + m(E_2) - m(E_3)} \cdots - m(E_N)} + m(E)$.

$\Rightarrow m(E_1) = m(E_1) + \lim_{N\to\infty}\left(-m(E_N)\right) + m(E)$

$\Rightarrow \lim_{N\to\infty} m(E_N) = m(E)$.

Thm: Any $E \in \mathcal{M}$ is well-approx by opens & closeds. I.e.

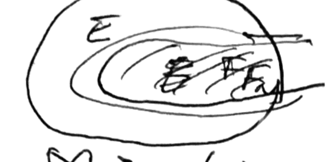
① $\forall \varepsilon > 0\ \exists \mathcal{O} \supset E$ s.t. $m(\mathcal{O} \setminus E) < \varepsilon$ ② $\exists F \subset E$ s.t. $m(E \setminus F) < \varepsilon$.

③ If $m(E) < \infty$. $\exists K$ cpt $\subset E$ s.t. $m(E \setminus K) < \varepsilon$.

④ If $m(E) < \infty$, $\exists$ closed cubes $Q_1, \dots, Q_N$. & $F = \bigcup_{j=1}^{N} Q_j$ with

$m(\underbrace{E \Delta F}_{E\setminus F \,\sqcup\, F\setminus E}) < \varepsilon$

pf ① def ② Apply to E^c ③ $\exists$ closed $F \subset E$ s.t. $m(E\setminus F) < \frac{\varepsilon}{2}$.



$F = \bigsqcup F\cap S_j$, $S_1 = \overline{B_1(0)}$, $S_k = \overline{B_k(0)} \setminus \overline{B_{k-1}(0)}$

$\infty > m(E) \geq m(F) = \underbrace{\sum_{j=1}^{\infty} m(F\cap S_j)}_{\text{converges}}$ $\Rightarrow \exists N$ with

$\sum_{j=N+1}^{\infty} m(F\cap S_j) < \frac{\varepsilon}{2}$. Let $F_N = \bigsqcup_{j=1}^{N} F\cap S_j = F\cap \overline{B_N(0)}$,

$m(E\setminus F_N) \le m(E) - m(F_N) \le \underbrace{m(E)-m(F)}_{\le \frac{\varepsilon}{2}} + \frac{\varepsilon}{2}$.

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pf ④: $\exists\, Q_1,\dots,$ closed cubes $E\subset \bigcup_{j=1}^{\infty} Q_j$ & $\infty > m(E) \ge \sum_{j=1}^{\infty} |Q_j| - \frac{\varepsilon}{2}$.

$\Rightarrow \exists N$ s.t. $\sum_{j=N+1}^{\infty} |Q_j| < \frac{\varepsilon}{2}$, $F = \bigcup_{j=1}^{N} Q_j$.

$m(E\Delta F) \le m(E\setminus F) + m(F\setminus E) \le \underbrace{m\Big(\bigcup_{j=N+1}^{\infty} Q_j\Big)}_{< \frac{\varepsilon}{2}} + \underbrace{m\Big(\bigcup_{j=1}^{\infty} Q_j \setminus E\Big)}_{< \frac{\varepsilon}{2}}$

Obvious facts: $m(E+h) = m(\{x+h \mid x\in E\}) = m(E)$.

$m(\delta E) = \delta^d m(E)$. $m(-E) = m(E)$.

Thm: $E\in\mathcal{M}$ (i) $\Leftrightarrow \exists\, G_\delta \supset E$ with $m(G_\delta\setminus E)=0$.

(ii) $\Leftrightarrow \exists\, F_\sigma \subset E$ with $m(E\setminus F_\sigma) = 0$.

where Def: $G_\delta = \bigcap_{j=1}^{\infty} O_j$ open $\in\mathcal{M}$ & $F_\sigma = \bigcup_{j=1}^{\infty} F_j$ closed $\in \mathcal{M}$

pf: (i)($\Leftarrow$) $G_\delta = E \cup (G_\delta\setminus E)$, $E = G_\delta \setminus (G_\delta\setminus E)$.

($\Rightarrow$) If $E\in\mathcal{M}$, $\forall n\ge 1$, $\exists\, O_n\supset E$ with $m(O_n\setminus E) \le \frac{1}{n}$.

Let $S = \bigcap_{n=1}^{\infty} O_n$. $m(S\setminus E) \le m(O_n\setminus E) \le \frac{1}{n} \to 0$. pf (ii) same

First measurable functions: characteristic functions

$$f(x) = \chi_E(x) = \begin{cases} 1 & x \in E \\ 0 & x \notin E, \end{cases} \qquad E \in \mathcal{M}.$$

Def: $f : \mathbb{R}^d \to \mathbb{R} \cup \{\pm\infty\}$ is measurable iff:

$\forall a \in \mathbb{R}$, $\{f < a\} = \{x \in E \mid f(x) < a\} = f^{-1}([-\infty, a)) \in \mathcal{M}$. $(-\infty, -n)$

Lemma: f finite valued is measurable $\iff \forall \mathcal{O} \subset \mathbb{R}$ open, $f^{-1}(\mathcal{O})$ is meas'ble. $\iff \forall F \subset \mathbb{R}$ closed, $f^{-1}(F)$ is meas'ble.

sketch: $\{f \leq a\} = \bigcap_{k=1}^{\infty} \{f < a + \frac{1}{k}\}$

& $\{f \geq a\}^c = \{f < a\}$ (& if $f^{-1}(\pm\infty) \in \mathcal{M}$.)

Lemma: If $f \in C(\mathbb{R}^d) \Rightarrow f$ meas'ble.

pf: Continuity $\iff f^{-1}(\mathcal{O}_{\text{open}}) = \mathcal{O}_{\text{open}} \in \mathcal{M}$.

Lemma: If f is measurable (& finite valued) & $\Phi(f(x)) : \mathbb{R}^d \to \mathbb{R}$, $f : \mathbb{R}^d \to \mathbb{R}$, & $\Phi : \mathbb{R} \to \mathbb{R}$ cont. Then $\Phi \circ f$ is meas'ble.

(Note: If $f : \mathbb{R} \to \mathbb{R}$ meas'ble & Φ cont, $f \circ \Phi$ not nec. meas'ble).

pf: Φ cont $\Rightarrow \Phi^{-1}((-\infty, a)) =$ open $\Rightarrow$

$(\Phi \circ f)^{-1}((-\infty, a)) = f^{-1}(\text{open}) \in \mathcal{M}$.

Lemma: $\{f_n\}$ meas, $\sup f_n(x) = f(x)$ is meas'ble.

pf: $\{\sup f_n > a\} = \bigcup_n \{f_n > a\} \in \mathcal{M}$.

Same gives: $\inf f_n$ is meas'ble.

$\limsup f_n$ & $\liminf f_n$ are meas'ble.

$$= \inf_{k \geq 1} \left(\sup_{n \geq k} f_n \right).$$