Lemmata & Thms & Def'n

Review: Def: $m_*(E^{\subset\mathbb{R}^d}) = \inf\limits_{E\subset\bigcup_{i=1}^{\infty} Q_i} \sum_{i=1}^{\infty} |Q_i|$.

Lemma 1: $m_*(E) = \inf\limits_{E\subset \mathcal{O}_{\text{open}}} m_*(\mathcal{O})$.

Lemma 2: $m_*\left(\bigcup_{j=1}^{\infty} E_j\right) \le \sum_{j=1}^{\infty} m_*(E_j)$

Lemma 3: If $\mathcal{O}$ = open, then $\mathcal{O} = \bigsqcup_{j=1}^{a.l.\,\infty} Q_j$.

Lemma 4: If $E = E_1 \sqcup E_2$ & $d(E_1, E_2) > 0$
$\Rightarrow m_*(E) = m_*(E_1) + m_*(E_2)$

Lemma 5: If $E = \bigsqcup_{j=1}^{a.l.\,\infty} Q_j \Rightarrow m_*(E) = \sum_{j=1}^{\infty} |Q_j|$.

Cor: $m_*(E)$ independent of cube decomposition.

Def: $E \subset \mathbb{R}^d$ is <u>measurable</u> if $\forall \varepsilon>0\ \exists\ \mathcal{O}_\varepsilon$ open $\supset E$

s.t. $m_*(\mathcal{O}_\varepsilon \setminus E) < \varepsilon$.

If E is measurable, then $\underline{m(E) = m_*(E)}$.

Def: $\mathcal{M} \subset \mathcal{P}(\mathbb{R}^d)$ be set of all measurable sets.

Lemma 6: $\mathcal{O}$ open $\Rightarrow \mathcal{O} \in \mathcal{M}$.

Building to: $\mathcal{M}$ is a σ-alg.
closed under complements & countable unions.

pf Lemma 4.(2) ~~let $\bigcup Q_j^{(1)} \supset E_1$ s.t. $\sum |Q_j^{(1)}| \le m_*(E_1) + \frac{\varepsilon}{2}$~~

~~& $\bigcup Q_j^{(2)} \supset E_2$.~~ Since $d(E_1, E_2) \ge \delta > 0$, can further subdivide Q_j to have diameter $< \delta$, & cover ~~Decompose~~ $E \subset \bigcup Q_j$, s.t. $\operatorname{diam}(Q_j) < \delta$ & $\sum |Q_j| \le m_*(E) + \varepsilon$.

so ~~Q_j~~ let $J_1 = \{ j \mid Q_j \cap E_1 \ne \emptyset \}$ & $J_2 = \mathbb{N} \setminus J_1$.

$\Rightarrow E_1 \subset \bigcup_{j \in J_1} Q_j$ & same E_2.

$$\Rightarrow m_*(E_1) + m_*(E_2) \le \sum_{j \in J_1} |Q_j| + \sum_{j \in J_2} |Q_j| \le m_*(E) + \varepsilon.$$

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pf Lemma 5: $m_*(E) \le \sum_{j=1}^{\infty} |Q_j|$. Need $(\ge)$.

For each j, let $\tilde{Q}_j \subset Q_j^{\circ}$ s.t. $|\tilde{Q}_j| \ge |Q_j| - \frac{\varepsilon}{2^j}$.

For any N, $\tilde{Q}_1, \dots, \tilde{Q}_N$ closed, disjoint & $d(\tilde{Q}_j, \tilde{Q}_k) > 0$.

(Lemma 4)

$$\Rightarrow m_*(E) \ge m_*\Big(\bigsqcup_{j=1}^{N} \tilde{Q}_j \Big) = \sum_{j=1}^{N} |\tilde{Q}_j| \ge \sum_{j=1}^{N} |Q_j| - \varepsilon$$

~~For~~ Send $N \to \infty$, $\varepsilon \to 0$.

What about $V = \mathbb{R}/\mathbb{Q}$.

More Lemmata; Etc

Abstract measure space $(X, \mathcal{M}, m)$.
$\mathcal{M} \subset \mathcal{P}(X)$

Lemma 7: If $m_*(E) = 0 \Rightarrow E$ is measurable. (Lemma 1).

Cor: Cantor set C is measurable. & $m(C) = 0$.

Lemma 8: Countable union of $E_j \in \mathcal{M}$ is $\in \mathcal{M}$.

~~pf: Let $E = \bigcup^{\infty} E_j$, $E_j \in \mathcal{M}$.~~

Lemma 9: E cpt $\Rightarrow E \in \mathcal{M}$.

Lemma 10: If K cpt & E closed & $K \cap E = \emptyset \Rightarrow d(K, E) > 0$.

Lemma 11: If E closed $\Rightarrow E \in \mathcal{M}$.

Lemma 12: If $E \in \mathcal{M} \Rightarrow E^c \in \mathcal{M}$.

Cor: $E_j \in \mathcal{M} \Rightarrow \bigcap_{j=1}^{\infty} E_j \in \mathcal{M}$.

Thm: If $E_j \in \mathcal{M}$ & pairwise disjoint $\Rightarrow E = \bigsqcup E_j \in \mathcal{M}$
& $m(E) = \sum_{j=1}^{\infty} m(E_j)$.

pf Lemma 8: $E = \bigcup E_j$, & $E_j \in \mathcal{M}$. Claim $E \in \mathcal{M}$.

For each E_j, $\exists\, \mathcal{O}_j$ with $m_*(\mathcal{O}_j \setminus E_j) < \frac{\varepsilon}{2^j}$.

$\mathcal{O} = \bigcup \mathcal{O}_j$ open $\supset E$. $m_*(\mathcal{O} \setminus E) \le \sum m_*(\mathcal{O}_j \setminus E_j) \le \varepsilon$.

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pf Lemma 9: E cpt, i.e. closed & bdd. By Lemma 1, $\exists\, \mathcal{O}$ open $\supset E$ s.t. $m_*(\mathcal{O}) \le m_*(E) + \varepsilon$. Obs:

$$\mathcal{O} \setminus E = \mathcal{O} \cap E^c = \text{open} = \bigsqcup_{j=1}^{\infty} Q_j \quad \text{(Lemma 3)}.$$

For fixed N, let $K_N = K = \bigsqcup_{j=1}^{N} Q_j$ (cpt). Claim: $d(K, E) > 0$.

By Lemma 10, claim true. $K \cup E \subset \mathcal{O} \overset{\text{(Lemma 4)}}{\Longrightarrow}$

$$m_*(E) + \varepsilon \ge m_*(\mathcal{O}) \ge m_*(K) + m_*(E) \ge m_*(E) + \sum_{j=1}^{N} m_*(Q_j).$$

$$m_*(\mathcal{O} \setminus E) \le \sum_{j=1}^{\infty} m_*(Q_j) \le m_*(\mathcal{O}) - m_*(E) \le \varepsilon.$$

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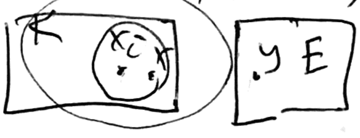
pf Lemma 10: $\forall x \in K$, $\exists\, \delta_x > 0$ s.t. $d(x, E) > 3\delta_x$.

$$K \subset \bigcup_x B_{2\delta_x}(x) \Rightarrow \exists\, x_1, \dots, x_N,\ K \subset \bigcup_{i=1}^{N} B_{3\delta_{x_i}}(x_i). \quad \delta = \min \delta_{x_i}$$

If $y \in E$, $x \in K$, ~~$d(y,x)$~~ $|y - x| = |y - x_i + x_i - x|$



$|y - x_i| > 3\delta$. $\ge |y - x_i| - |x_i - x| > 3\delta - 2\delta \ge \delta$.

pf Lemma 11: $E = \bigcup_{k=1}^{\infty} \underbrace{E \cap \overline{B_k(0)}}_{\text{cpt} \Rightarrow \in \mathcal{M}} \xRightarrow{(\text{Lemma } 8)} E \in \mathcal{M}$.

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pf Lemma 12: Since E is meas'ble, $\exists\, O_n$ s.t.

$m_*(O_n \setminus E) \le \frac{1}{n}$. $\Rightarrow$ O_n^c is closed $\Rightarrow \in \mathcal{M}$,

& $S := \bigcup_{n=1}^{\infty} O_n^c \in \mathcal{M}$. Obs: $\underbrace{E^c \setminus S}_{= E^c \cap S^c} \subset O_n \setminus E$.

$S^c = \bigcap O_n$

$\Rightarrow$ ~~[illegible]~~ $m_*(E^c \setminus S) \le \frac{1}{n} \to 0$. $\Rightarrow E^c \setminus S \in \mathcal{M}$.

$\Rightarrow E^c = E^c \setminus S \cup S \in \mathcal{M}$
