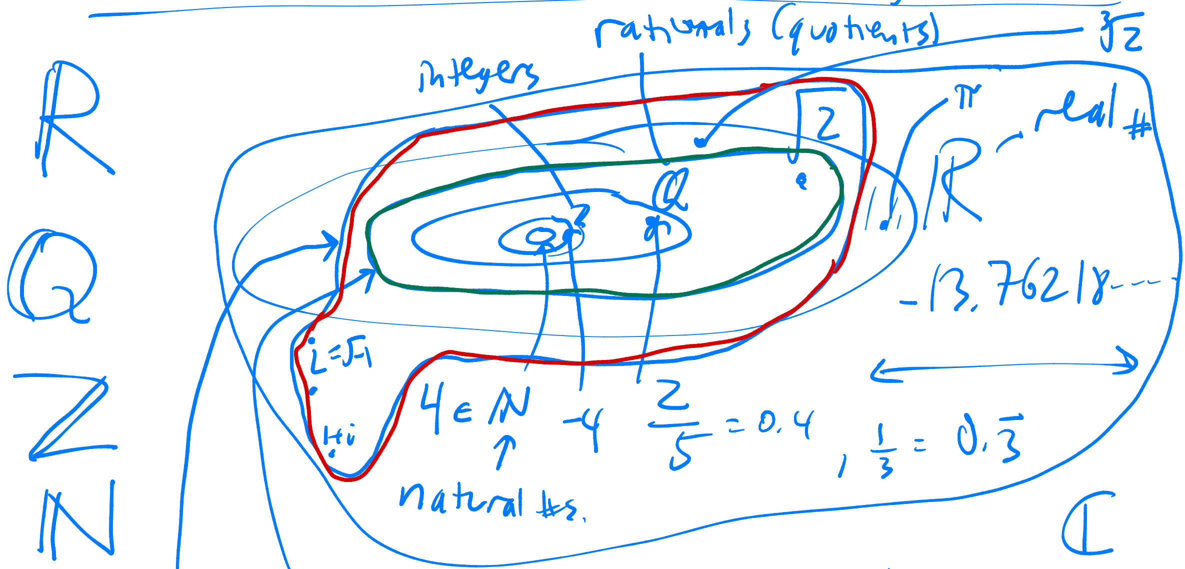
Last time: Resolution of Construction Problems From Antiquity: I Trisecting an angle II Squaring circle III Doubling cube.



Complex

$\mathbb{B}$ ← "Constructible numbers".

$\mathbb{A}$ ← "algebraic numbers".

Def: α is a root of $f(x)$ if $f(\alpha) = 0$.

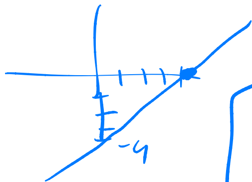
Def: A number is "algebraic" if it is the root (zero) of a polynomial with integer coefficients

$0 \cdot x^8 + 14x^7 - 3x^3 + 2x - 1$.

$a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0$

Q: Is $4 \in A$? $f(x) = \cancel{x}^1 - 4$.

$f(4) = 0$.



$\boxed{x = \frac{2}{5}, \quad 5x = 2, \quad 5x - 2 = 0.}$

Q: Is $\frac{2}{5} \in A$? $f(x) = \cancel{x - \frac{2}{5}}$. $\notin \mathbb{Z}$.

$f(x) = 5x - 2$. $f(\frac{2}{5}) = 0$.

Q: Is $\sqrt{2} \in A$? $f(x) = x^2 - 2$.

$f(\sqrt{2}) = (\sqrt{2})^2 - 2 = 2 - 2 = 0$.

$x^2 - 2 = 0, \quad x^2 = 2, \quad x = \sqrt{2}, -\sqrt{2}$.

Q: Is $i \in A$? $f(x) = 1x^2 + 1$.

$x^2 + 1 = 0, \quad x^2 = -1, \quad x = i, -i$.

Q: Is $\sqrt{2 + \sqrt[4]{3}} \in A$? ✓

$x^8 - 19$. $(2 + 3^{1/4})^{1/2} = x$

$2+3^{1/4}=x^2$, $3^{1/4}=x^2-2$.

$(x^2-2)^4=3$, $\boxed{f(x)=(x^2-2)^4-3}$,

$f(x)=x^8-8x^6+24x^4-32x^2+\cancel{16-3}$
13.

Def: A real number $\alpha\in\mathbb{B}$ (ie is constructable) if it can be expressed using only square-roots, $+,-,\times,\div$ & $\boxed{\frac{1}{3}-\sqrt{\frac{2}{\sqrt{3}}+\frac{\sqrt{4}}{5}}}$

$\boxed{\text{Q: Is } \sqrt{2}\in\mathbb{B}\text{? Yes.}}$

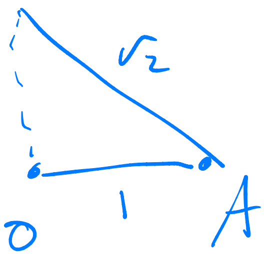
Q: Is $\sqrt{2+\sqrt[4]{3}}\in\mathbb{B}$? Yes.

$\parallel$

$\sqrt{2+\sqrt{\sqrt{3}}}$ $\qquad (3^{1/2})^{1/2}=3^{\frac{1}{2}\cdot\frac{1}{2}}=3^{\frac{1}{4}}=\sqrt[4]{3}$

Q: Is $\sqrt[3]{2}\in\mathbb{B}$? Ans: NO.

Thm (Gauss 1797): If a number $\alpha \in \mathbb{R}$ is constructible, then there is a geometric (straight edge and compass) process that creates it.



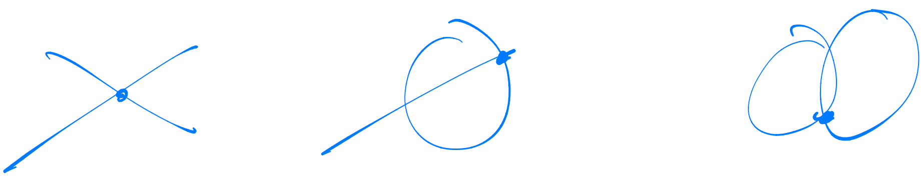
Call given length $\overline{OA}$ = unit.

Can you geom construct $\underline{\sqrt{2}}$?

Thm (Pierre Wantzel 1837): And conversely.

That is, the *only* lengths that can be created by straightedge and compass are constructible.

Hint: How are new points created? By intersection of lines + lines OR lines + circles OR circles + circles

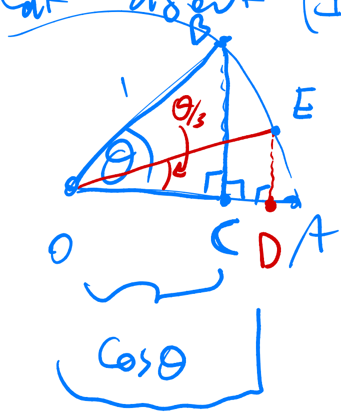


Algebraically what's happening when you solve for these new points ONLY ever involves at most taking a square root of other lengths already constructed.

This already resolves the question (I) of doubling a cube: it can't be done, because

$$\sqrt[3]{2} \notin \mathbb{B}.$$

What about (II) Trisecting an angle? ($\overline{OA} = 1$)



The act of creating angle theta is the same as creating the length OC.

$$\overline{OC} = \frac{\overline{OC}}{1 = \overline{OB}} = \frac{adj}{hyp} = \cos\theta.$$

So trisecting theta is the same as creating OD = $\cos\frac{\theta}{3} = x$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

$$\cos(2\alpha) = \cos^2\alpha - \sin^2\alpha = 2\cos^2\alpha - 1.$$

$$\cos(3\alpha) = 4\cos^3\alpha - 3\cos\alpha.$$

$$\alpha = \frac{\theta}{3}.$$

$$\overline{OC} = \cos\theta = 4\cdot\cos^3\left(\frac{\theta}{3}\right) - 3\cos\left(\frac{\theta}{3}\right)$$

$$= \boxed{4x^3 - 3x = \overline{OC}}.$$

↑ known.

This equation is $\underline{\text{cubic}}$.

general cubic equas. need $\sqrt[3]{\ }$ somewhere in solution.

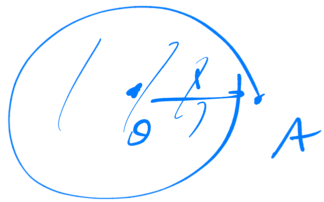
The equation for cos(theta/3) (in terms of known cos (theta)) is a cubic equation. Its solution requires cube roots, which cannot be obtained by square roots. So cos (theta /3) is (almost always) not constructible.

Therefore we've solved (II) - in the negative, that is, an arbitrary angle CANNOT be trisected with straightedge and compass.

So what about (III) Squaring the Circle?

$r=1$

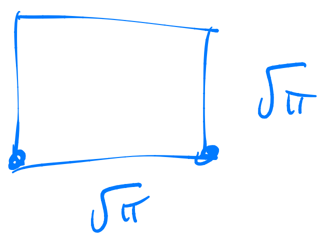
$\text{Area} = \pi \cdot r^2 = \pi$



Want: length $\sqrt{\pi}$.

π.

Q: What kind of polynomial has sqrt pi as its root?



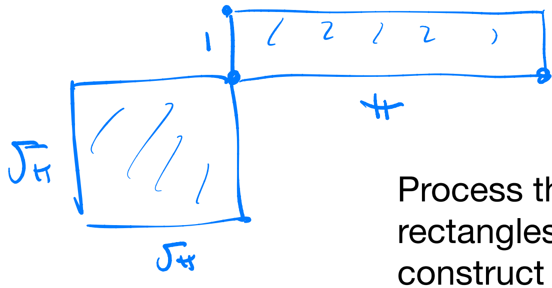
Answer: NONE!!!! Pi is "transcendental".
I.e. For any polynomial f(x) with integer coeffiecients, f(pi) $\neq 0$.

There is no polynomial with π as a root.

The fact that pi is transcendental was proved by Lindemann in 1882

(Following important work of Hermite.)

Not done! why not $\sqrt{\pi}$?



Process that proves the quadrature of rectangles, if run "in reverse", would construct a 1 x ? rectangle with the same area as sqrt pi x sqrt pi square. So ? = pi.

Can't square a circle!!!

Hippocrates 450 BCE

Euclid 350 BCE ~ 300 BCE.

Alexander the Great. 350 BCE

Founds Alexandria.

City becomes an epicenter of math.

Euclid's Elements. 300 BCE.