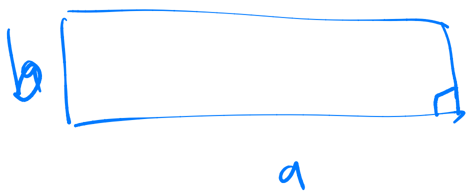
<u>Last time:</u> Proved Quadrature of <u>ALL</u> <u>Rectilinear Shapes</u> (i.e. polygons)

How could <u>you</u> have discovered these constructions on your own???

Rectangle:



Want area: $\underline{\underline{a \cdot b}}$, side length $\sqrt{ab}$.

<u>Notice:</u>

$$(a+b)^2 = \cancel{a^2} + 2ab + \cancel{b^2}.$$

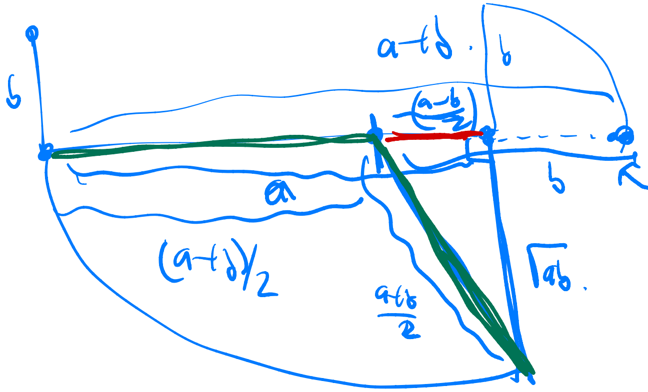
$$-(a-b)^2 = -\cancel{a^2} + 2ab \mp \cancel{b^2}.$$

$$(a+b)^2 - (a-b)^2 = 4ab.$$

Divide both sides by 4,

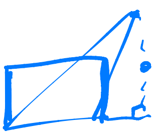
$$\left(\frac{a+b}{2}\right)^2 - \left(\frac{a-b}{2}\right)^2 = ab = \left(\sqrt{ab}\right)^2.$$

$$\left(\frac{a+b}{2}\right)^2 = \left(\frac{a-b}{2}\right)^2 + \left(\sqrt{ab}\right)^2.$$



$$\text{Red} = \frac{a}{2} + \frac{b}{2} - b.$$
$$= \frac{a}{2} - \frac{b}{2}.$$

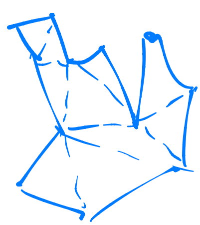
Quad of Rect $\Rightarrow$ Quad of Triangle



$$\text{Area(Triangle)} = \frac{1}{2}\ \text{base} \times \text{height}.$$

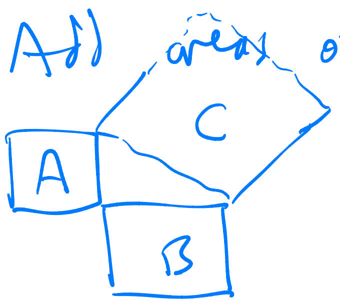
Then any Rectilinear Shape (i.e Polygon)

is Quadrable (Squarable).

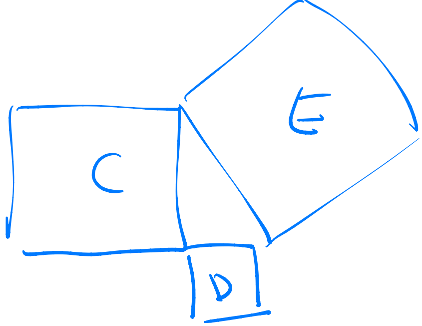


Two Ideas: ① Triangulate

② Add areas of squares. (Pyth Thm).



$C = A + B$ (as areas).

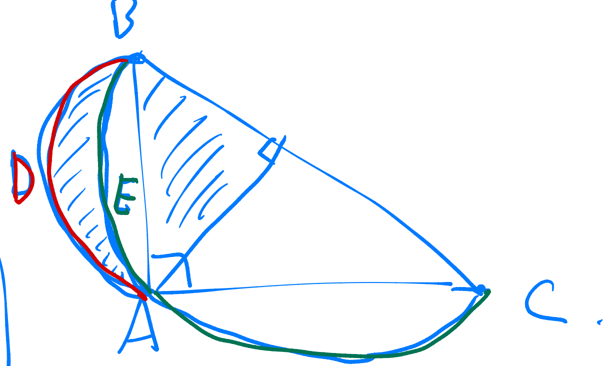


$E = C + D.$

Great Thm: Hippocrates's Quadrature of the Lune. 450-400 BCE.

Setup: ① Right isosceles triangle.

② Draw semicircle on BC (E)
(Converse to Thales, if ⊾ then semicircle on diameter goes thru)

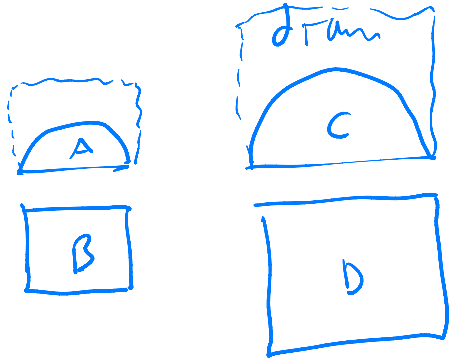


③ Draw semicircle on AB (D).

Thm: ADBE = Lune is Quadable.
(This is curvilinear = bounded by arcs of circles).

pf: Uses FACT:

$$\frac{\text{Area (semicircle)}}{\text{Area (square on diam)}} = \text{Constant.}$$



FACT: $\frac{\text{Area}(A)}{\text{Area}(B)} = \frac{\text{Area}(C)}{\text{Area}(D)}$

$\qquad\ \ \|\qquad\qquad\quad \|$

$\qquad\ \frac{\pi}{8}\qquad\qquad\ \ \frac{\pi}{8}.$

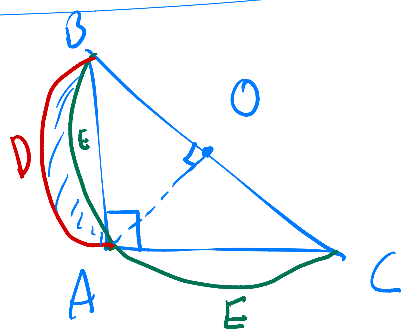
Note: dispute among historians on whether Hippocratus really knew how to prove this fact. It does appear (with proof) in Euclid's Elements...

In modern terms, $\text{Area(circle)} = \pi r^2$.

$\text{Area(semicircle)} = \frac{1}{2}\pi r^2$. $\quad \text{Diam} = 2r$.

$\text{Area(square)} = (2r)^2 = 4 \cdot r^2$

$$\frac{\text{Area (semicircle)}}{\text{Area(square)}} = \frac{\frac{1}{2}\pi \cancel{r^2}}{4 \cdot \cancel{r^2}} = \frac{\pi}{8} = \underline{\text{Const}}.$$



$\overline{AB}^2 + \overline{AC}^2 = \overline{BC}^2$

$2\overline{AB}^2 = \overline{BC}^2$.

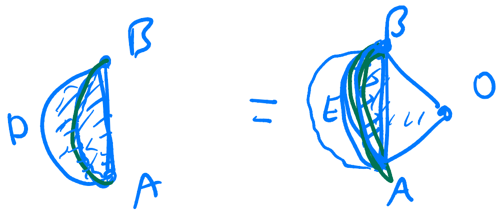
$\overline{AB}^2 = \frac{1}{2} \cdot \overline{BC}^2$.

Pf: Look at Area(semi ABD) = $\frac{1}{2}$ Area(semi BCE)

= Area(quarter BOAE).

① Bisect BC at O, draw AO

Again



Area(semi BAD) − Area(ABE) = Area(quar BOAE) − Area(ABE),

Area(lune) = Area(△BOA).

But triangles are quadrable!

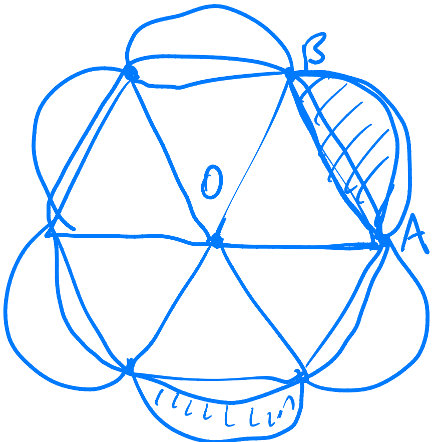
Hence so is the lune QED.

Supposedly, Hippocrates then claimed that he could square the circle.

" (FALSE) "

Claim: The circle is quadrable.

pf: ① Construct regular hexagon



② Draw ⊙ c O r AO

③ Draw semicircle on $\overline{AB}$ & 5 more

radius $= \frac{1}{2}\overline{AB}$

Look at total area:

Big circle + 6 Lunes = hexagon + 6 semicircles.

rad(Big circle) = 2 rad(little).

$\Rightarrow$ Area(Big circle) = 4 Area(little circle).

↳ 4 Area(little circle) + 6 Lunes = Hex + 3 Area(little circles)

⇒

Area(Circle on $\overline{AB}$) = Hex − 6 Lunes.

quadrable quadrable

⇒ ~~Circles are quadrable~~

Hippocrates did find two other lunes that really were quadrable. (Not this one) Leonhard Euler 1771 found two more. 20th century: Tchebotarev and Dorodnow proved that these five lunes really are the only ones that are quadrable.

Big Three Construction Problems of Antiquity

① Trisecting an angle

② Doubling a cube

③ Squaring a circle.

Punchline: None are solvable.

Wantzel ① & ② 1837 Thm.

③: Lindemann 1885.