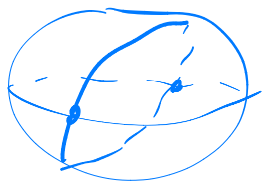
Last time:

Euclid's geometry is "synthetic", that is, axiomatic, "points" "lines" and "planes" (should be) are undefined terms, the postulates say what you can do with them. P1: any two points determine a line.

1868 Beltrami: realizes the role of "models" in showing that P5 cannot be proved from the other four.

What's happening in Spot It? Every pair of cards has exactly one icon in common. Should satisfy: Every pair of icons appear on a unique card.

Spot It is a "model" for (Finite) Projective Geometry



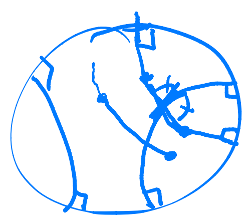
"Plane" = game, "Line" = icon, "Point" = card OR alternatively, "Line" = card, "Point" = icons.

Beltrami makes a "model" for the geometry "discovered" by Lobachevsky, Bolyai, Gauss.

Poincare 1882 gave a slightly different version.

"plane" = disk "point" = point in disk

"line" = circular arc orthogonal to boundary

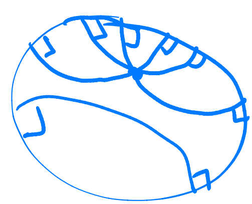


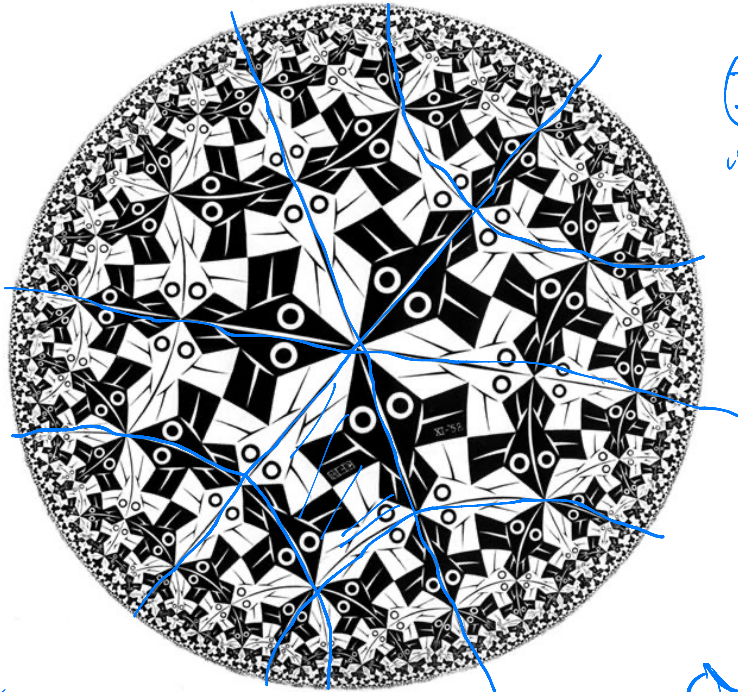
P1: Draw line (segment) between two points

P2: Continue line segment a little further

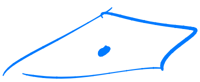
P3: Draw circle (r).

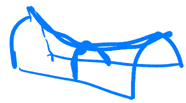
P4: All ∠'s are equal.



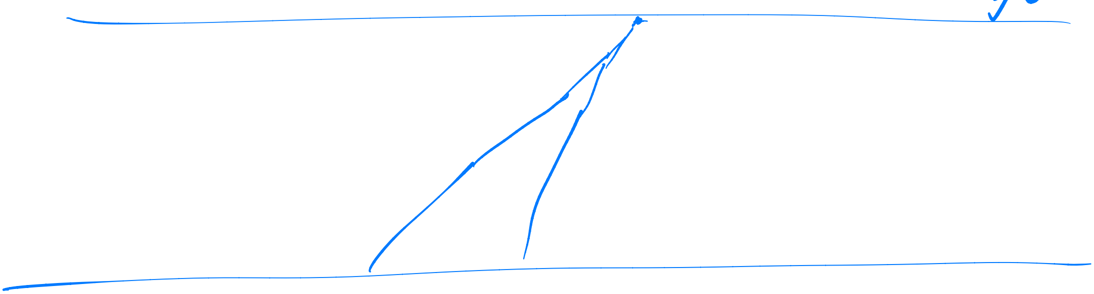


Escher "Circle limits"? 1958? Coxeter.





hyperbolic geometry.



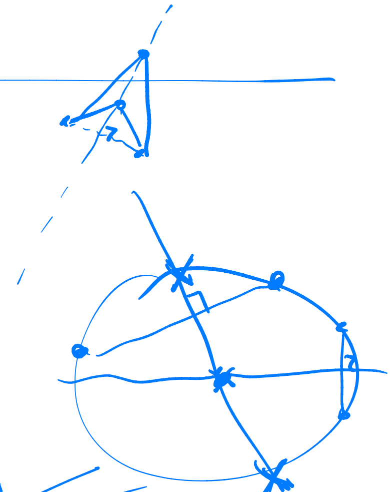
How does Beltrami conclude that P5 CANNOT be proved from the other four? Any FACT in Euclidean geometry, is also a FACT in hyperbolic geometry, just with the undefined terms "point" "line" "circle" "plane" being shuffled around from one model to the other. So any potential contradiction in hyperbolic geometry has a corresponding statement in Euclidean geometry that is also a contradiction!!! So it's not possible to prove P5 since it's not true in hyperbolic geometry while P1-P4 are.

~~Axioms~~ → Thm → Thm → Thm → ~~Fact~~

(P1-P4) + (~~¬P5~~) →
Not

Book II : geometric proofs of things like Prop 4:
(a+b)^2 = a^2 + 2ab +b^2

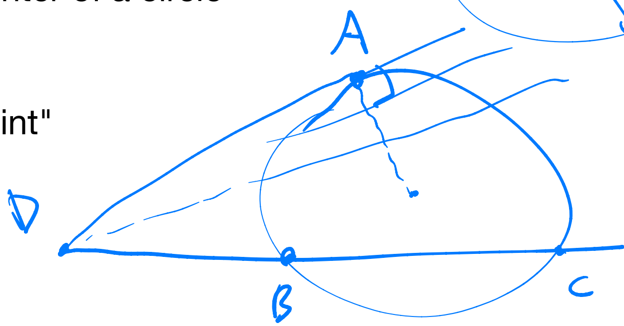
Prop 14: Quadrature of the Rectangle



Book III: Circles, tangents, ...

Prop III.1: To find the center of a circle

Prop III.36: "Power of a point"

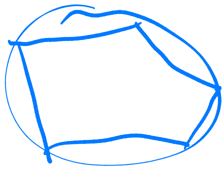


$$\overline{DB} \cdot \overline{DC} = \overline{DA}^2$$

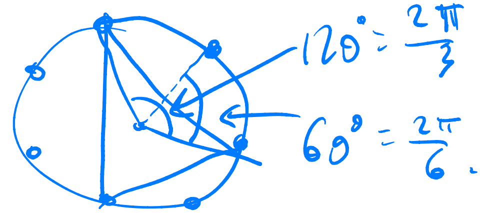
Prop III.37: converse : if BD * DC = DA^2 holds, then DA is tangent to the circle

Book IV: inscribe a circle, circumscribe a circle...
Prop 16 ("Big boss") : construct the 15-gon!!

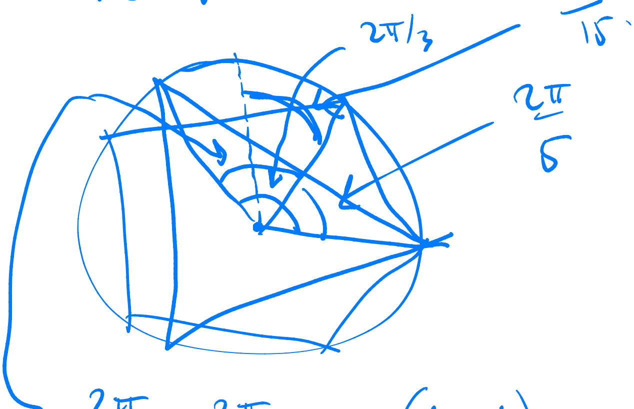
regular 3-gon: I.1 (triangle), 4-gon: I.46 (square), 5-gon: IV.13.



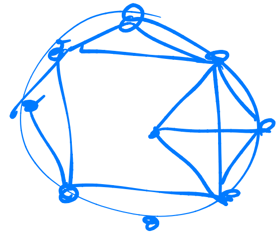
6-gon: IV.15.



15-gon: IV.16.



$$\frac{2\pi}{3} - \frac{2\pi}{5} = 2\pi\left(\frac{1}{3} - \frac{1}{5}\right)$$
$$= 2\pi\left(\frac{2}{15}\right)$$



By bisecting, you can make:

3-gon, 6-gon, 12, 24, ... $2^n \cdot 3$-gon,

4 ... 2^n-gon

5-gon, 10, 20, 40 ... $5 \cdot 2^n$ ...

15, 30, 60, ... $15 \cdot 2^n$.

These constructions of regular n-gons were the only ones known for 2000 years.
in 1796, 19 year old named... Carl Friedrich Gauss, discovers that the 17-gon is constructible!!!

The reason the 17-gon is constructible, is that 17=16+1.
3=2+1, 5=4+1,

Gauss's theorem: n-gon (for n prime) is constructible if and only if:
n = Fermat prime

Fermat (1630s) discovered: 2^n + 1 can only have a chance of being prime if n itself is a power of 2. Really:

$$2^{2^m}+1 = p.$$

$$1+2^{2^0} = 3$$

$$1+2^{2^1} = 5$$

$$1+2^{2^2} = 17.$$

$$1+2^{2^3} = 257$$

$$1+2^{2^4} = 65537$$

all prime. But all the other powers computed so far have all turned out to be composite!!