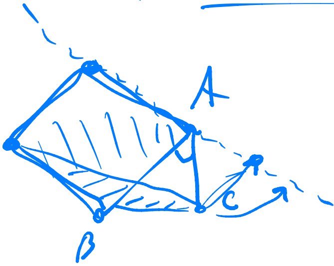
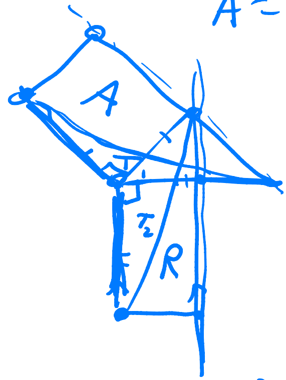
History of Math, Princeton University, Fall 2024, Prof. Kontorovich

Last time: I.47.





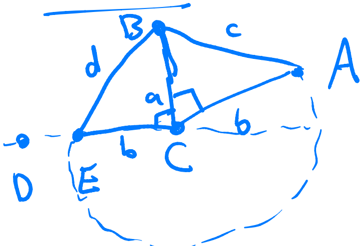
$A = 2T_1$

$T_1 = T_2$

$R = 2T_2$

$\Rightarrow A = R$.

I.48 Converse: If $a^2+b^2=c^2 \Rightarrow \angle C = \square$.



pf: ① Make $DC \perp BC$ (I.11)

② Draw $\odot_C$ r CA, let E (P3) on DC with $\overline{EC} = \overline{AC} = b$

③ Draw BE (P1), $\overline{BE} = d$.

By I.47 $\Rightarrow a^2+b^2 = d^2 \overset{(CN1)}{=} c^2$, $\Rightarrow c = d$.

$\Rightarrow$ (SSS) $\triangle ABC \cong \triangle EBC$ $\Rightarrow$ $\angle$'s correspond.

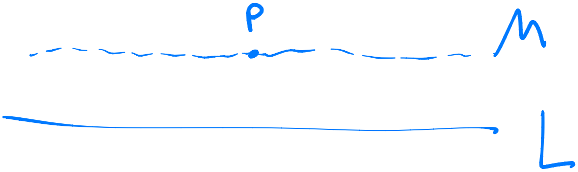
$\Rightarrow \underbrace{\angle BCE}_{= \square \text{ (by construction)}} = \angle BCA \underset{(CN1)}{=} \square$.

QED.

For the next 2000 years, people are debating / arguing etc about whether the 5th postulate really "belongs" as an axiom (aesthetically), or (mathematically), could it be proved already from the first four.

dozens of alternative axioms were proposed to replace P5.

Equivalent: given a line and a point, there exists a unique parallel through the point



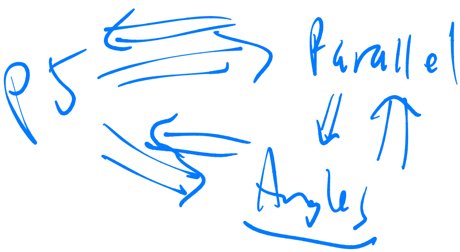
Given L = line & P = point, $\exists!$ (there exists a unique) line $M \parallel L$.

Note: whether two lines are indeed parallel is NOT a property that can be checked in finite time.

Using the parallel postulate, gives a mechanism by which you indeed can verify the condition: e.g., drop a transverse line and check if opposite angles agree.

What does having such an alternative AXIOM/POSTULATE mean? Using P5, you can prove Parallel Axiom, and conversely.

Another alternative: the sum of angles in any triangle is 180.



If you could assume that there exists some triangle with sum of angles not 180, using only P1 - P4, and prove a CONTRADICTION. Then you would show that P5 is indeed a THEOREM and not an axiom.

So lots of people tried to make the assumption that there exists a single triangle with sum of angles not 180, and see if they could follow the logic, writing some strange alternative version of the Elements, until they hopefully arrive at a contradiction.

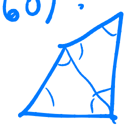
What if sum of angles is more than 180. (Turns out that if there exists a single triangle with sum of angles = 180, then you get the parallel postulate) This would hold for all possible triangles.
One alternative world: All triangles have angle sum > 180.

Gauss 1810's (?) starts playing with this alternative universe, and under the extra assumption that lines are infinitely long, manages to arrive at a contradiction.

Remains: Other alternative: all triangles have angle sum < 180.
Bolyai.

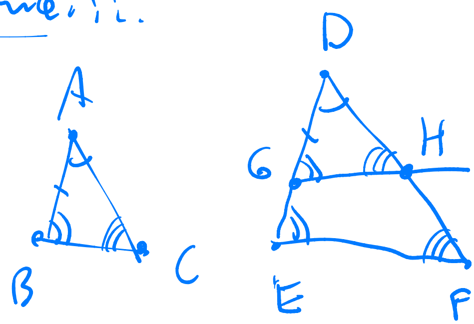
Sample theorem of Bolyai:

Assume that any triangle has angle sum <180 ($\Rightarrow$ angle sum in quadrilateral <360).



Thm: AAA $\Rightarrow$ Congruence!!!

pf: Given ΔABC, ΔDEF, $\measuredangle A = \measuredangle D$, $\measuredangle B = \measuredangle E$, $\measuredangle C = \measuredangle F$.



Assume $\overline{AB} < \overline{DE}$.

① By I.3, draw G on DE with $\overline{AB} = \overline{DG}$.

② By I.7(??), can move angles, draw $\measuredangle$ at $G = \measuredangle B$.

By ASA, $\Delta ABC \cong \Delta DGH \Rightarrow \measuredangle H = \measuredangle C$

$\Rightarrow$ angle sum in quad GHFE $= 360°$. X

Bolyai continues in this way, failing to find a contradiction, but constructing this crazy new world where the parallel postulate is false.

1832, Bolyai publishes a treatise of this crazy new geometry. Sends this manuscript to Gauss for comment.
Gauss: I cannot praise this work, for to praise it would be to praise myself!
Nobody knew that Gauss was working on this problem, in fact, he wrote letters to friends 1824, says to his friend that he won't ever publish this work: "I fear the howls if I speak my mind out loud".

1829 Lobachevsky discovered and published many of the same theorems.

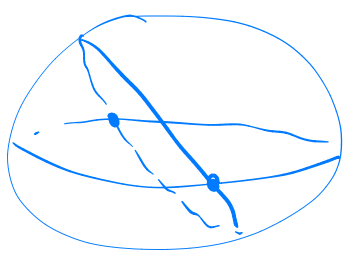
This geometry is now called hyperbolic geometry.

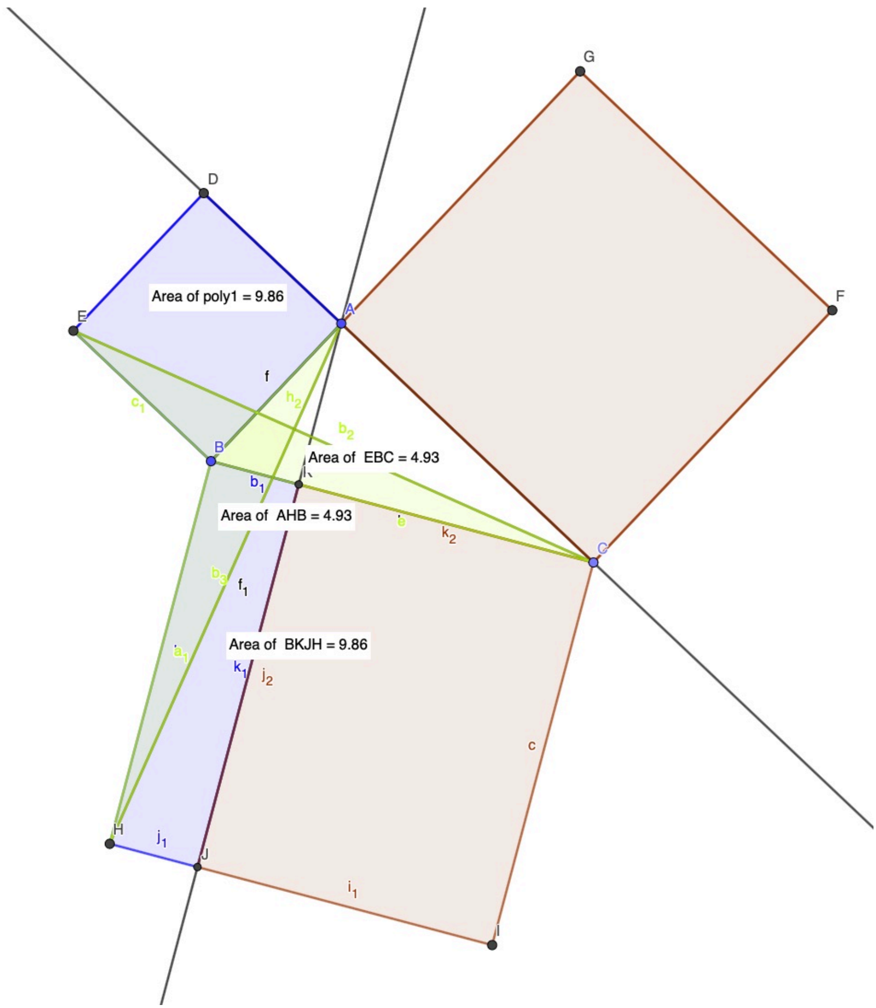
Riemann 1855 returns to sum of angles being more than 180 degrees, lines merely need to be unbounded (P2), not infinitely long. Should not have been ruled out.

Beltrami 1868 finally puts the issue to bed, proving that the 5th postulate cannot be proved from the other four.

Develops idea of "models" for geometry. Goes all the way back to definitions of "plane", "points", "lines", "circles"... How do we model what's going on in synthetic geometry? (Euclid gives pure axioms for what you're allowed to do, should have been agnostic about a model that realizes those axioms.)

E.g.: Model for Euclidean space? Rene Descrates (1630s) gives us the Cartesian plane, (x,y), for real numbers x,y, as model for the Euclidean plane. "Line" ax+by=c. "Point" (x,y). "Circle" = circle.
Another model for "Plane" = surface of sphere. "Line" = great circle. "Point" = pair of antipodal points on the surface of the sphere. ("Projective geometry")





G
D
F
Area of poly1 = 9.86
A
E
f
c1
h2
b2
B
Area of EBC = 4.93
K
b1
Area of AHB = 4.93
e
k2
C
b3
f1
Area of BKJH = 9.86
a1
k1
j2
c
H
j1
J
i1
I