FOURIER ANALYSIS IN NUMBER FIELDS

AND HECKE'S ZETA-FUNCTIONS

By

JOHN TORRANCE TATE, JR.

A DISSERTATION

Presented to the

Faculty of Princeton University

in Candidacy for the Degree

of Doctor of Philosophy

Recommended for Acceptance by the

Department of

Mathematics

May, 1950.

Abstract.

We lay the foundations for abstract analysis in the groups of valuation vectors and idèles associated with a number field. This allows us to replace the classical notion of ζ-function, as the sum over integral ideals of a certain type of ideal character, by the corresponding notion for idèles, namely, the integral over the idèle group of a rather general weight function times an idèle character which is trivial on field elements. The role of Hecke's complicated theta - formulas for theta functions formed over a lattice in the n-dimensional space of classical number theory can be played by a simple Poisson Formula for general functions of valuation vectors, summed over the discrete subgroup of field elements. With this Poisson Formula, which is of great importance in itself, inasmuch as it is the number theoretic analogue of the Riemann-Roch theorem, an analytic continuation can be given at one stroke for all of the generalized ζ-functions, and an elegant functional equation can be established for them. Translating these results back into classical terms one obtains the Hecke functional equation, together with an interpretation of the complicated factor in it as a product of certain local factors coming from the archimedean primes and the primes of the conductor. The notion of local ζ-function has been introduced to give a local definition of these factors, and a table of them has been computed.

CONTENTS.

CHAPTER I.

Introduction.

1.1 Relevant History. Hecke was the first to prove that the Dedekind ζ-function of any algebraic number field has an analytic continuation over the whole plane and satisfies a simple functional equation. He soon realized that his method would work, not only for the Dedekind ζ-function and L-series, but also for a ζ-function formed with a new type of ideal character which, for principal ideals depends not only on the residue class of the number modulo the "conductor", but also on the position of the conjugates of the number in the complex field. Overcoming rather extraordinary technical complications, he showed (1) that these "Hecke" ζ-functions satisfied the same type of functional equation as the Dedekind ζ-function, but with a much more complicated factor.

In a work (2) the main purpose of which was to take analysis out of class field theory, Chevalley introduced the excellent notion of the idèle group, as a refinement of the ideal group. In idèles Chevalley had not only found the best approach to class field theory, but to algebraic number theory generally. This is shown by Artin and Whaples in (3). They defined valuation vectors as the additive counterpart of idèles, and used these notions to derive from simple axioms all of the basic statements of algebraic number theory.

Matchett, a student of Artin's, made a first attempt (4) to continue this program and do analytic number theory by means of idèles and vectors. She succeeded in redefining the classical ζ-functions in terms of integrals over the idèle group, and in interpreting the characters of Hecke as exactly those characters of the ideal group which can be derived from idèle characters. But in proving the functional equation she followed Hecke.

1.2 This Thesis. Artin suggested to me the possibility of generalizing the notion of ζ-function, and simplifying the proof of the analytic continuation and functional equation for it, by making fuller use of analysis in the spaces of valuation vectors and idèles themselves than Matchett had done. This thesis is the result of my work on his suggestion. I replace the classical notion of ζ-function, as the sum over integral ideals of a certain type of ideal character, by the corresponding notion for idèles, namely, the integral over the idèle group of a rather general weight function times an idèle character which is trivial on field elements. The role of Hecke's complicated theta-formulas for theta functions formed over a lattice in the n-dimensional space of classical number theory can be played by a simple Poisson Formula for general functions of valuation vectors, summed over the discrete subgroup of field elements. With this Poisson Formula, which is of great importance in itself, inasmuch as it is the number theoretic analogue of the Riemann-Roch Theorem, an analytic continuation can be given at one stroke for all of the generalized ζ-functions, and an elegant functional equation can be established for them. Translating these results back into classical terms one obtains the Hecke functional equation, together with an interpretation of the complicated factor in it as a product of certain local factors coming from the archimedean primes and the primes of the conductor. The notion of local ζ-function has been introduced to give a local definition of these factors, and a table of them has been computed.

I wish to express to Artin my great appreciation for his suggestion of this topic and for the continued encouragement he has given me in my work.

<u>1.3 "Prerequisites"</u>. In number theory we assume only the knowledge of the classical algebraic number theory, and its relation to the local theory. No knowledge of the idèle and valuation vector point of view is required, because, in order to introduce abstract analysis on the idèle and vector

groups we redefine them and discuss their structure in detail.

Concerning analysis, we assume only the most elementary facts and definitions in the theory of analytic functions of a complex variable. No knowledge whatsoever of classical analytic number theory is required. Instead, the reader must know the basic facts of abstract Fourier analysis in a locally compact abelian group G: 1.) The existence and uniqueness of a Haar measure on such a group, and its equivalence with a positive invariant functional on the space $L(G)$ of continuous functions on G which vanish outside a compact. 2.) The duality between G and its character group, $\widehat{G}$, and between subgroups of G and factor groups of $\widehat{G}$. 3.) The definition of the Fourier transform, $\widehat{f}$, of a function $f \in L_1(G)$, together with the fact that, if we choose in $\widehat{G}$ the measure which is dual to the measure in G, the Fourier Inversion Formula holds (in the naive sense) for all functions for which it could be expected to hold; namely, for functions $f \in L_1(G)$ such that f is continuous and $\widehat{f} \in L_1(\widehat{G})$. (This class of functions we denote by $\mathcal{V}_1(G)$). An elegant account of this theory can be found for example in (5).

C H A P T E R 2.

The Local Theory.

2.1 Introduction. Throughout this section, k denotes the completion of an algebraic number field at a prime divisor $\mathfrak{p}$. Accordingly, k is either the real or complex field if $\mathfrak{p}$ is archimedean, while k is a "$\mathfrak{p}$-adic" field if $\mathfrak{p}$ is discrete. In the latter case k contains a ring of integers $\mathfrak{o}$ having a single prime ideal $\mathfrak{p}$ with a finite residue class field $\mathfrak{o}/\mathfrak{p}$ of $N\mathfrak{p}$ elements. In both cases k is a complete topological field in the topology associated with the prime divisor $\mathfrak{p}$.

From the infinity of equivalent valuations of k belonging to $\mathfrak{p}$ we select the normed valuation defined by:

$|\alpha|$ = ordinary absolute value if k is real.

$|\alpha|$ = square of ordinary absolute value if k is complex.

$|\alpha|$ = $(N\mathfrak{p})^{-\nu}$, where ν is the ordinal number of α, if k is $\mathfrak{p}$-adic.

We know that k is locally compact. The more exact statement which one can prove is: a subset $B \subset k$ is relatively compact (has a compact closure) if and only if it is bounded in absolute value. Indeed, this is a well known fact for subsets of the line or plane if k is the real or complex field; and one can prove it in a similar manner in case k is $\mathfrak{p}$-adic by using a "Schubfachschluss" involving the finiteness of the residue class field.

2.2 Additive Characters and Measure. Denote by k^+ the additive group of k, as a locally compact commutative group, and by ξ its general element. We wish to determine the character group of k^+, and are happy to see that this task is essentially accomplished by the following:

Lemma 2.2.1: If $\xi \to X(\xi)$ is one non-trivial character of k^+, then for each $\eta \in k$, $\xi \to X(\eta\xi)$ is also a character. The correspondence $\eta \leftrightarrow X(\eta\xi)$

is an isomorphism, both topological and algebraic, between k^+ and its character group.

Proof: 1.) $X(\eta\xi)$ is a character for any fixed η because the map $\xi \to \eta\xi$ is a continuous homomorphism of k^+ into itself.

2.) $X((\eta_1+\eta_2)\xi) = X(\eta_1\xi+\eta_2\xi) = X(\eta_1\xi)X(\eta_2\xi)$ shows that the map $\eta \to X(\eta\xi)$ is an algebraic homomorphism of k^+ into its character group.

3.) $X(\eta\xi) = 1$, all $\xi \Rightarrow \eta k^+ \neq k^+ \Rightarrow \eta = 0$. Hence it is an algebraic isomorphism into.

4.) $X(\eta\xi) = 1$, all $\eta \Rightarrow k^+\xi \neq k^+ \Rightarrow \xi = 0$. Therefore the characters of the form $X(\eta\xi)$ are everywhere dense in the character group.

5.) Denote by B the (compact) set of all $\xi \in k$ with $|\xi| \leq M$ for a large M. Then: η close to 0 in $k^+ \Rightarrow \eta B$ close to 0 in $k^+ \Rightarrow X(\eta B)$ close to 1 in complex plane $\Rightarrow X(\eta\xi)$ close to the identity character in the character group. On the other hand, if ξ_0 is a fixed element with $X(\xi_0) \neq 1$, then: $X(\eta\xi)$ close to identity character $\Rightarrow X(\eta B)$ close to 1, closer, say, than $X(\xi_0) \Longrightarrow \xi_0 \notin B \Longrightarrow \xi_0$ close to 0 in k^+. Therefore the correspondence $\eta \longleftrightarrow X(\eta\xi)$ is bicontinuous.

6.) Hence the characters of the form $X(\eta\xi)$ comprise a locally compact subgroup of the character group. Local compactness ~~implies compactness~~ implies completeness and therefore closure, which together with 4.) shows that the mapping is onto.

To fix the identification of k^+ with its character group promised by the preceding lemma, we must construct a special non-trivial character. Let p be the rational prime divisor which $\mathfrak{p}$ divides, and R the completion of the rational field at p. Define a map $x \to \lambda(x)$ of R into the reals mod 1 as follows:

Case 1.) $\mathfrak{p}$ archimedean, and therefore R the real numbers.

$$\lambda(x) \equiv -x \pmod 1$$

(Note the minus sign!)

Case 2.) $\mathfrak{p}$ discrete, R the field of p-adic numbers. $\lambda(x)$ shall be determined by the properties:

a.) $\lambda(x)$ is a rational number with only a p-power in the denominator.

b.) $\lambda(x) - x$ is a p-adic integer.

(To find such a $\lambda(x)$, let $p^{\nu}x$ be integral, and chose an ordinary integer n such that $n \equiv p^{\nu}x \pmod{p^{\nu}}$. Then put $\lambda(x) = n/p^{\nu}$; $\lambda(x)$ is obviously uniquely determined modulo 1.)

Lemma 2.2.2: $x \to \lambda(x)$ is a non-trivial, continuous additive map of R into the group of reals (mod 1).

Proof In case 1.) this is trivial. In case 2.) we check that the number $\lambda(x) + \lambda(y)$ satisfies properties a) and b) for $x+y$, so the map is additive. It is continuous at 0, yet non-trivial because of the obvious property: $\lambda(x) \equiv 0 \Leftrightarrow x$ is p-adic integer.

Define now for $\xi \in k^+$, $\Lambda(\xi) = \lambda(S_{k/R}\xi)$. Recalling that $S_{k/R}$ is an additive continuous map of k onto R, we see that $\xi \to e^{2\pi i \Lambda(\xi)}$ is a non-trivial character of k^+. We have proved:

Theorem 2.21: k^+ is naturally its own character group if we identify the character $\xi \to e^{2\pi i \Lambda(\eta\xi)}$ with the element $\eta \in k^+$.

Lemma 2.23: In case $\mathfrak{p}$ is discrete, the character $\xi \to e^{2\pi i \Lambda(\eta\xi)}$ associated with η is trivial on $\mathfrak{o}$ if and only if $\eta \in \mathfrak{d}^{-1}$, $\mathfrak{d}$ denoting the absolute different of k.

Proof: $\Lambda(\eta\mathfrak{o}) = 0 \Leftrightarrow \lambda(S_{k/R}(\eta\mathfrak{o})) \equiv 0 \Leftrightarrow S_{k/R}(\eta\mathfrak{o}) \subset \mathfrak{o}_R$.

Let now μ be a Haar measure for k^+.

<u>Lemma 2.2.4</u>: If we define $\mu_1(M) = \mu(\alpha M)$ for $\alpha \neq 0 \in k$, and M a measurable set in k^+, then μ_1 is a Haar measure, and consequently there exists a number $\varphi(\alpha) > 0$ such that $\mu_1 = \varphi(\alpha)\mu$.

<u>Proof</u>: $\xi \to \alpha\xi$ is an automorphism of k^+, both topological and algebraic. Haar measure is determined, up to a positive constant, by the topological and algebraic structure of k^+.

<u>Lemma 2.2.5</u>: The constant $\varphi(\alpha)$ of the preceding lemma is $|\alpha|$, i.e. we have $\mu(\alpha M) = |\alpha| \mu(M)$.

<u>Proof</u>: If k is the real field, this is obvious. If k is complex, it is just as obvious since in that case we chose $|\alpha|$ to be the square of the ordinary absolute value. If k is $\mathfrak{p}$-adic, we notice that since $\mathfrak{o}$ is both compact and open, $0 < \mu(\mathfrak{o}) < \infty$, and it therefore suffices to compare the size of $\mathfrak{o}$ with that of $\alpha\mathfrak{o}$. For α integral, there are $N(\alpha\mathfrak{o})$ cosets of $\alpha\mathfrak{o}$ in $\mathfrak{o}$, hence $\mu(\alpha\mathfrak{o}) = (N(\alpha\mathfrak{o}))^{-1}\mu(\mathfrak{o}) = |\alpha| \mu(\mathfrak{o})$. For non-integral α, replace α by α^{-1}.

We have now another reason for calling the normed valuation the natural one. $|\alpha|$ may be interpreted as the factor by which the additive group k^+ is "stretched" under the transformation $\xi \to \alpha\xi$.

For the integral, the meaning of the preceding lemma is clearly: $d\mu(\alpha\xi) = |\alpha| d\mu(\xi)$; or more fully: $\int f(\xi)\, d\mu(\xi) = |\alpha| \int f(\alpha\xi)\, d\mu(\xi)$.

So much for a general Haar measure μ. Let us now select a fixed Haar measure for our additive group k^+. Theorem 2.2.1 enables us to do this in an invariant way by selecting that measure which is its own Fourier transform under the interpretation of k^+ as its own character group established in that theorem. We state the choice of measure which does this, writing $d\xi$ instead of $d\mu(\xi)$, for simplicity:

$d\xi$ = ordinary Lesbegue measure on real line if k is real.

$d\xi$ = twice ordinary Lesbegue measure in the plane if k is complex.

$d\xi$ = that measure for which $\mathfrak{o}$ gets measure $(N\mathfrak{d})^{-\frac{1}{2}}$ if k is $\mathfrak{p}$-adic.

Theorem 2.2.2: If we define the Fourier transform $\hat{f}$ of a function $f \in L_1(k^+)$ by:

$$\hat{f}(\eta) = \int f(\xi)\, e^{-2\pi i \Lambda(\eta\xi)}\, d\xi\,,$$

then with our choice of measure, the inversion formula

$$f(\xi) = \int \hat{f}(\eta)\, e^{+2\pi i \Lambda(\eta\xi)} d\eta = \hat{\hat{f}}(-\xi)$$

holds for $f \in \mathcal{V}_1(k^+)$.

Proof: We need only establish the inversion formula for one non-trivial function, since from abstract Fourier analysis we know it is true, save possibly for a constant factor. For k real we can take $f(\xi) = e^{-\pi|\xi|^2}$, for k complex, $f(\xi) = e^{-2\pi|\xi|}$; and for k $\mathfrak{p}$-adic, $f(\xi)$ = the characteristic function of $\mathfrak{o}$, for instance. For the details of the computations, the reader is referred to §2.6 below.

2.3 Multiplicative Characters and Measure. Our first insight into the structure of the multiplicative group $k^\times$ of k is given by the continuous homomorphism $\alpha \to |\alpha|$ of $k^\times$ into the multiplicative group of positive real numbers. The kernel of this homomorphism, the subgroup of all α with $|\alpha| = 1$ will obviously play an important role. Let us denote it by u. u is compact in all cases, and in case k is $\mathfrak{p}$-adic, u is also open.

Concerning the characters of $k^\times$, the situation is different from that of k^+. First of all, we are interested in all continuous multiplicative maps $\alpha \to c(\alpha)$ of $k^\times$ into the complex numbers, not only in the bounded ones, and shall call such a map a quasi-character, reserving the word "character" for the conventional character of absolute value 1. Secondly, we shall find no model for the group of quasi-characters, or even for the group of characters, though such a model would be of the utmost importance.

We call a quasi-character unramified if it is trivial on $\mathfrak{u}$, and first determine the unramified quasi-characters.

Lemma 2.3.1: The unramified quasi-characters are the maps of the form $c(\alpha) = |\alpha|^s = e^{s \log |\alpha|}$, where s is any complex number, s is determined by c if $\mathfrak{p}$ is archimedean, while for discrete $\mathfrak{p}$, s is determined only mod $2\pi i/\log N\mathfrak{p}$.

Proof: For any s, $|\alpha|^s$ is obviously an unramified quasi-character. On the other hand any unramified quasi-character will depend only on $|\alpha|$, and as function of $|\alpha|$ will be a quasi-character of the value group of k. This value group is the multiplicative group of all positive real numbers, or of all powers of $N\mathfrak{p}$, according to whether $\mathfrak{p}$ is archimedean or discrete; it is well known that the quasi-characters of these groups are those described.

If $\mathfrak{p}$ is archimedean, we may write the general element $\alpha \in k^\times$ uniquely in the form $\alpha = \tilde{\alpha}\rho$, with $\tilde{\alpha} \in \mathfrak{u}$, $\rho > 0$. For discrete $\mathfrak{p}$, we must select a fixed element π of ordinal number 1 in order to write, again uniquely, $\alpha = \tilde{\alpha}\rho$, with $\tilde{\alpha} \in \mathfrak{u}$ and, this time, ρ a power of π. In either case the map $\alpha \to \tilde{\alpha}$ is a continuous homomorphism of $k^\times$ onto $\mathfrak{u}$ which is identity on $\mathfrak{u}$.

Theorem 2.3.1: The quasi-characters of $k^\times$ are the maps of the form $\alpha \to c(\alpha) = \tilde{c}(\tilde{\alpha})\,|\alpha|^s$, where $\tilde{c}$ is any character of $\mathfrak{u}$. $\tilde{c}$ is uniquely determined by c. s is determined as in the preceding lemma.

Proof: A map of the given type is obviously a quasi-character. Conversely, if c is a given quasi-character and we define $\tilde{c}$ to be the restriction of c to u, then $\tilde{c}$ is a quasi-character of u and is therefore a character of u since u is compact. $\alpha \to c(\alpha)/\tilde{c}(\tilde{\alpha})$ $\cancel{\Longleftrightarrow}$ is an unramified quasi-character, and therefore is of the form $|\alpha|^s$ according to the preceding lemma.

The problem of quasi-characters c of $k^\times$ therefore boils down to that of the characters $\tilde{c}$ of u. If k is the real field, $u = \{1, -1\}$ and the characters are $\tilde{c}(\tilde{\alpha}) = \tilde{\alpha}^m$, $m = 0, 1$. If k is complex, u is the unit-circle, and the characters are $\tilde{c}(\tilde{\alpha}) = \tilde{\alpha}^m$, n any integer. In case k is $\mathfrak{p}$-adic, the subgroups $1+\mathfrak{p}^\nu$, $\nu>0$, of u form a fundamental system of neighborhoods of 1 in u. We must have therefore $\tilde{c}(1+\mathfrak{p}^\nu) = 1$ for sufficiently large ν. Selecting ν minimal ($\nu = 0$ if $\tilde{c} = 1$), we call the ideal $\mathfrak{f} = \mathfrak{p}^\nu$ the conductor of $\tilde{c}$. Then $\tilde{c}$ is a character of the finite factor group $(u/1+\mathfrak{f})$ and may be described by a finite table of data.

From the expression $c(\alpha) = \tilde{c}(\tilde{\alpha})\,|\alpha|^s$ for the general quasi-character given in theorem 2.3.1, we see that $|\,c(\alpha)\,| = |\alpha|^\sigma$, where $\sigma = \mathrm{Re}(s)$ is uniquely determined by $c(\alpha)$. It will be convenient to call σ the exponent of c. A quasi-character is a character if and only if its exponent is 0.

We will be able to select a Haar measure $d\alpha$ on $k^\times$ by relating it to the measure $d\xi$ on k^+. If $g(\alpha) \in \mathrm{L}(k^\times)$, then $g(\xi)\,|\xi|^{-1} \in \mathrm{L}(k^+ - 0)$. So we may define on $\mathrm{L}(k^\times)$ a functional

$$\Phi(g) = \int_{k^+-0} g(\xi)\,|\xi|^{-1} d\xi .$$

If $h(\alpha) = g(\beta\alpha)$ $\quad(\beta \in k^\times$, fixed) is a multiplicative translation of

$g(\alpha)$, then

$$\Phi(h) = \int_{k^+-0} g(\beta\xi)\,|\xi|^{-1} d\xi = \Phi(g),$$

as we see by the substitution $\xi \to \beta^{-1}\xi$; $d\xi \to |\beta|\,d\xi$ discussed in lemma 2.2.5. Therefore our functional Φ which is obviously non-trivial and positive, is also invariant under translation. It must therefore come from a Haar measure on $k^{\times}$. Denoting this measure by $d_1\alpha$, we may write

$$\int g(\alpha)\, d_1\alpha = \int_{k^+-0} g(\xi)\,|\xi|^{-1} d\xi .$$

Obviously, the correspondence $g(\alpha) \leftrightarrow g(\xi)\,|\xi|^{-1}$ is a 1-1 correspondence between $L(k^{\times})$ and $L(k^+-0)$. Viewing the functions of $L_1(k^{\times})$ and $L_1(k^+-0)$ as limits of these basic functions we obtain:

<u>Lemma 2.3.2</u>: $g(\alpha) \in L_1(k^{\times}) \Leftrightarrow g(\xi)\,|\xi|^{-1} \in L_1(k^+-0)$, and for these functions

$$\int g(\alpha)\, d_1\alpha = \int_{k^+-0} g(\xi)\,|\xi|^{-1} d\xi .$$

For later use, we need a multiplicative measure which will in general give the subgroup $\mathfrak{u}$ the measure 1. To this effect we choose as our standard Haar measure on $k^{\times}$:

$$d\alpha = d_1\alpha = \frac{d\alpha}{|\alpha|}, \text{ if } \mathfrak{p} \text{ is archimedean.}$$

$$d\alpha = \frac{N\mathfrak{p}}{N\mathfrak{p}-1}\, d_1\alpha = \frac{N\mathfrak{p}}{N\mathfrak{p}-1}\,\frac{d\alpha}{|\alpha|}, \text{ if } \mathfrak{p} \text{ is discrete.}$$

<u>Lemma 2.3.3</u>: In case $\mathfrak{p}$ is discrete, $\int_{\mathfrak{u}} d\alpha = (N\mathfrak{d})^{-\frac{1}{2}}$.

<u>Proof</u>: $\int_{\mathfrak{u}} d_1\alpha = \int_{\mathfrak{u}} |\xi|^{-1} d\xi = \int_{\mathfrak{u}} d\xi = \frac{N\mathfrak{p}-1}{N\mathfrak{p}} \int_{\mathfrak{o}} d\xi$. Therefore

$$\int_{\mathfrak{u}} d\alpha = \frac{N\mathfrak{p}}{N\mathfrak{p}-1} \int_{\mathfrak{u}} d_1\alpha = \int_{\mathfrak{o}} d\xi = (N\mathfrak{d})^{-\frac{1}{2}}.$$

2.4. <u>The Local ζ-function; Functional Equation</u>: In this section $f(\xi)$ will denote a complex valued function defined on k^+ ; $f(\alpha)$ its restriction to $k^{\times}$. We let $\mathfrak{z}$ denote the class of all these functions which satisfy the two conditions:

$\mathfrak{Z}_1$) $f(\xi)$, and $\hat{f}(\xi)$ continuous, $\in L_1(k^+)$; i.e. $f(\xi) \in \mathfrak{W}_1(k^+)$

$\mathfrak{Z}_2$) $f(\alpha)|\alpha|^\sigma$ and $\hat{f}(\alpha)|\alpha|^\sigma \in L_1(k^\times)$ for $\sigma > 0$.

A ζ-function of k will be what one might call a multiplicative quasi-Fourier transform of a function $f \in \mathfrak{Z}$. Precisely what we mean is stated in

<u>Definition 2.4.1</u>: Corresponding to each $f \in \mathfrak{Z}$, we introduce a function $\zeta(f,c)$ of quasi-characters c, defined for all quasi-characters of exponent greater than 0 by

$$\zeta(f,c) = \int f(\alpha)c(\alpha)\, d^\times\alpha,$$

and call such a function a ζ-function of k.

Let us call two quasi-characters equivalent if their quotient is an unramified quasi-character. According to lemma 2.3.1, an equivalence class of quasi-characters consists of all quasi-characters of the form $c(\alpha) = c_0(\alpha)|\alpha|^s$, where $c_0(\alpha)$ is a fixed representative of the class, s a complex variable. It is apparent that by introducing the complex parameter s we may view an equivalence class of quasi-characters as a Riemann surface. In case $\mathfrak{p}$ is archimedean, s is uniquely determined by c, and the surface will be isomorphic to the complex plane. In case $\mathfrak{p}$ is discrete, s is determined only mod $2\pi i/\log N\mathfrak{p}$, so the surface is isomorphic to a complex plane in which points differing by an integral multiple of $2\pi i/\log N\mathfrak{p}$ are identified - the type of surface on which singly periodic functions are really defined. Looking at the set of all quasi-characters as a collection of Riemann surfaces, it becomes clear what we mean when we talk of the regularity of a function of quasi-characters at a point or in a region, or of singularities. We may also consider the question of analytic continuation of such a function, though this must of course be carried out on each surface (equivalence class of quasi-characters) separately.

Lemma 2.4.1: A ζ-function is regular in the "domain" of all quasi-characters of exponent greater than 0.

Proof: We must show that for each c of exponent 0 the integral $\int f(\alpha) c(\alpha)|\alpha|^s d^\times\alpha$ represents a regular function of s for s near 0. Using the fact that the integral is absolutely convergent for s near 0 to make estimates, it is a routine matter to show that the function has a derivative for s near 0. The derivative can in fact be computed by "differentiating under the integral sign".

It is our aim to show that the ζ-functions have a single-valued meromorphic analytic continuation to the domain of all quasi-characters by means of a simple functional equation. We start out from

Lemma 2.4.2: For c in the domain $0 <$ exponent $c < 1$ and $\hat{c}(\alpha) = |\alpha| \;\; c^{-1}(\alpha)$ we have

$$\zeta(f,c)\,\zeta(\hat{g},\hat{c}) = \zeta(\hat{f},\hat{c})\;\;\zeta(g,c)$$

for any two functions $f,g \in \mathfrak{Z}$.

Proof: $\zeta(f,c)\;\;\zeta(\hat{g},\hat{c}) = \int f(\alpha)c(\alpha)d^\times\alpha \cdot \int \hat{g}(\beta)\;c^{-1}(\beta)\;|\beta|\,d^\times\beta$ with both integrals absolutely convergent for c in the region we are considering. We may write this as an absolutely convergent "double integral" over the direct product, $k^\times \times k^\times$ of $k^\times$ with itself:

$$\iint f(\alpha)\;\hat{g}\;(\beta)\;c\;(\alpha\beta^{-1})\;|\beta|\;d^\times(\alpha,\beta).$$

Subjecting $k^\times \times k^\times$ to the "shearing" automorphism $(\alpha,\beta) \to (\alpha,\alpha\beta)$, under which the measure $d^\times(\alpha,\beta)$ is invariant we obtain

$$\iint f(\alpha)\;\hat{g}(\alpha\beta)\;c(\beta^{-1})\;|\alpha\beta|\;d^\times(\alpha,\beta)\;.$$

According to Fubini this is equal to the repeated integral

$$\int\left(\int f(\alpha)\;\;\hat{g}\;\;(\alpha\beta)\;|\alpha|\,d^\times\alpha\right)c(\beta^{-1})\;|\beta|\,d^\times\beta\;.$$

To prove our contention it suffices to show that the inner integral $\int f(\alpha)\hat{g}(\alpha\beta)\;|\alpha|\,d^\times\alpha$ is symmetric in f and g. This we do by writing down the obviously symmetric additive double integral:

$$\iint f(\xi)g(\eta) \quad e^{-2\pi i \Lambda(\xi\beta\eta)}\, d(\xi,\eta),$$

changing it with the Fubini theorem into

$$\int f(\xi) \left(\int g(\eta) e^{-2\pi i \Lambda(\xi\beta\eta)}\, d\eta\right) d\xi = \int f(\xi)\hat{g}(\xi\beta)\, d\xi,$$ and

observing that according to lemma 2.3.2 this last expression is equal to the multiplicative integral

$$\int f(\alpha)\hat{g}(\alpha\beta)|\alpha|\, d_1\alpha = \text{constant}\cdot\int f(\alpha)\hat{g}(\alpha\beta)|\alpha|\, d\alpha$$

We can now announce the Main Theorem of the local theory.

Theorem 2.4.1: A ζ-function has an analytic continuation to the domain of all quasi-characters, given by a functional equation of the type

$$\zeta(f,c) = \rho(c)\ \zeta(\hat{f},\hat{c}).$$

The factor $\rho(c)$, which is independent of the function f, is a meromorphic function of quasi-characters defined in the domain $0 <$ exponent $c < 1$ by the functional equation itself, and for all quasi-characters by analytic continuation.

Proof: In the next section we will exhibit for each equivalence class C of quasi-characters an explicit function $f_C \in \mathfrak{Z}$ such that the function

$$\rho(c) = \frac{\zeta(f_C, c)}{\zeta(\hat{f}_C, \hat{c})}$$

is defined (i.e. has denominator not identically 0) for c in the strip $0 <$ exponent $c < 1$ on C. The function $\rho(c)$ defined in this manner will turn out to be a familiar meromorphic function of the parameter s with which we describe the surface C, and therefore will have an analytic continuation over all of C.

From these facts, which will be proved in §2.5 the theorem follows directly. For since C was any equivalence class, $\rho(c)$ is defined for all quasi-characters. And if $f(\xi)$ is any function of $\mathfrak{Z}$ we have according to the preceding lemma

$$\zeta(f,c)\ \zeta(\hat{f}_C,\hat{c}) = \zeta(\hat{f},\hat{c})\ \zeta(f_C,c),$$

therefore $$\zeta(f,c) = \rho(c)\ \zeta(\hat{f},\hat{c}),$$

if c is any quasi-character in the domain of $0 <$ exponent $c<1$, where $\zeta(f,c)$ and $\zeta(\hat{f},\hat{c})$ are originally both defined, and C is the equivalence class of c.

Before going on to the computations of the next section which will put this theory on a sound basis, we can prove some simple properties of the factor $\rho(c)$ in the functional equation which follow directly from the functional equation itself.

<u>Lemma 2.4.3:</u>

$$1.)\ \rho(\hat{c}) = \frac{c(-1)}{\rho(c)}. \qquad 2.)\ \rho(\bar{c}) = c(-1)\ \overline{\rho(c)}.$$

<u>Proof:</u>

$$1.)\ \zeta(f,c) = \rho(c)\ \zeta(\hat{f},\hat{c}) = \rho(c)\rho(\hat{c})\ \zeta(\hat{\hat{f}},\hat{\hat{c}}) = c(-1)\ \zeta(f,c)$$

because $\hat{\hat{f}}(\alpha) = f(-\alpha)$ and $\hat{\hat{c}}(\alpha) = c(\alpha)$. Therefore $\rho(c)\rho(\hat{c}) = c(-1)$.

$$2.)\ \overline{\zeta(f,c)} = \zeta(\bar{f},\bar{c}) = \rho(\bar{c})\ \zeta(\hat{\bar{f}},\hat{\bar{c}})$$

$$= \rho(\bar{c})\ c(-1)\ \zeta(\bar{\hat{f}},\bar{\hat{c}}) = \rho(\bar{c})\ c(-1)\ \overline{\zeta(\hat{f},\hat{c})}$$

because $\hat{\bar{f}}(\alpha) = \hat{f}(-\alpha)$ and $\hat{\bar{c}}(\alpha) = \bar{\hat{c}}(\alpha)$. On the other hand

$$\overline{\zeta(f,c)} = \overline{\rho(c)}\ \ \overline{\zeta(\hat{f},\hat{c})}.$$

Therefore $$\rho(\bar{c}) = c(-1)\ \overline{\rho(c)}.$$

<u>Corollary 2.4.1:</u> $|\rho(c)| = 1$ for c of exponent $\frac{1}{2}$.

<u>Proof:</u> (exponent c) $= \frac{1}{2} \Rightarrow c(\alpha)\ \bar{c}(\alpha) = |c(\alpha)|^2 = |\alpha| = c(\alpha)\hat{c}(\alpha) \Rightarrow$

$\bar{c}(\alpha) = \hat{c}(\alpha)$. Equating the two expressions for $\rho(\bar{c})$ and $\rho(\hat{c})$ given in the preceding lemma yields $\rho(c)\ \overline{\rho(c)} = 1$.

2.5 <u>Computation of $\rho(c)$ by Special ζ-functions.</u> This section contains the computations promised in the proof of theorem 2.4.1. For each equivalence class C of quasi-characters we give an especially simple function $f_C \in \mathfrak{Z}$ with which it is easy to compute $\rho(c)$ on the surface C. Carrying out this computation we obtain a table which gives the analytic expression for $\rho(c)$ in terms of the parameter, s, on each surface C. It will be necessary to treat the cases k real; k complex; and k $\mathfrak{p}$-adic separately.

k real.

ξ is a real variable. α is a non-zero real variable.

$\Lambda(\xi) = -\xi$ $|\alpha|$ is the ordinary absolute value.

$d\xi$ means ordinary Lesbegue measure. $d^{\times}\alpha = \dfrac{d\alpha}{|\alpha|}$

<u>The Equivalence Classes of Quasi-Characters.</u> The quasi-characters of the form $|\alpha|^s$, which we denote simply by $|\ |^s$, comprise one equivalence class. Those of the form $(\text{sign}\,\alpha)|\alpha|^s$, which we denote by $\pm|\ |^s$, comprise the other.

<u>The Corresponding Functions of $\mathfrak{z}$:</u> We put

$$f(\xi) = e^{-\pi\xi^2} \quad \text{and} \quad f_{\pm}(\xi) = \xi e^{-\pi\xi^2}.$$

<u>Their Fourier Transforms:</u> We contend

$$\hat{f}(\xi) = f(\xi) \quad \text{and} \quad \hat{f}_{\pm}(\xi) = i f_{\pm}(\xi).$$

Indeed, these are simply the two identities

$$\int_{-\infty}^{\infty} e^{-\pi\eta^2+2\pi i\xi\eta}\, d\eta = e^{-\pi\xi^2} \quad \text{and} \quad \int_{-\infty}^{\infty} \eta e^{-\pi\eta^2+2\pi i\xi\eta}\, d\eta = i\xi e^{-\pi\xi^2}$$

familiar from classical Fourier analysis. The first of these can be established directly by completing the square in the exponent, making the complex substitution $\eta \to \eta + i\xi$, which is allowed by Cauchy's integral theorem, and replacing the definite integral $\int_{-\infty}^{\infty} e^{-\pi\xi^2}d\xi$ by its well-known value 1. The second identity is obtained by applying the operation $\dfrac{1}{2\pi i}\ \dfrac{d}{d\xi}$ to the first.

<u>The ζ-functions:</u> We readily compute:

$$\zeta(f, |\ |^s) = \int f(\alpha)\, |\alpha|^s d^{\times}\alpha = \int_{-\infty}^{\infty} e^{-\pi\alpha^2}\, |\alpha|^s\, \frac{d\alpha}{|\alpha|}$$

$$= 2\int_0^{\infty} e^{-\pi\alpha^2}\alpha^{s-1}\, d\alpha = \pi^{-\frac{s}{2}}\, \Gamma\left(\frac{s}{2}\right)$$

$$\zeta(f_\pm, \pm|\ |^s) = \int f_\pm(\alpha)(\pm|\alpha|^s)d\alpha = \int_{-\infty}^{0} \alpha e^{-\pi\alpha^2}(-1)|\alpha|^s \frac{d\alpha}{|\alpha|} + \int_0^\infty x e^{-\pi x^2}|x|^s \frac{dx}{|x|}$$
$$= 2\int_0^\infty e^{-\pi\alpha^2}\alpha^s d\alpha = \pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right).$$
$$\zeta(\hat{f}, \widehat{|\ |^s}) = \zeta(f, |\ |^{1-s}) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)$$
$$\zeta(\hat{f}_\pm, \widehat{\pm|\ |^s}) = \zeta(if_\pm, \pm|\ |^{1-s}) = i\pi^{-\frac{(1-s)+1}{2}}\Gamma\left(\frac{(1-s)+1}{2}\right).$$

Explicit Expressions for $\rho(c)$:

$$\rho(|\ |^s) = \frac{\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)}{\pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)} = 2^{1-s}\pi^{-s}\cos\left(\frac{s\pi}{2}\right)\Gamma(s)$$

$$\rho(\pm|\ |^s) = -i\frac{\pi^{-\frac{s+1}{2}}\Gamma\left(\frac{s+1}{2}\right)}{\pi^{-\frac{(1-s)+1}{2}}\Gamma\left(\frac{(1-s)+1}{2}\right)} = -i\,2^{1-s}\pi^{-s}\sin\left(\frac{s\pi}{2}\right)\Gamma(s)$$

Here the quotient expressions for ρ come directly from the definition of ρ as quotient of suitable ζ-functions; the second form follows from elementary Γ-function identities.

k Complex.

$\xi = x + iy$ is a complex variable.

$\Lambda(\xi) = -2\,\mathrm{Re}(\xi) = -2x.$

$d\xi = 2\,|dxdy|$ is twice the ordinary Lebesgue measure.

$\alpha = re^{i\theta}$ is a non-zero complex variable.

$\|\alpha\| = r^2$ is ~~twice~~ the square of the ordinary absolute value.

$$d\alpha = \frac{d\alpha}{\|\alpha\|} = \frac{2r\,|dr\,d\theta|}{r^2} = \frac{2}{r}|dr\,d\theta|.$$

Equivalence Classes of Quasi-Characters: The characters $c_n(\alpha)$ defined by $c_n(re^{i\theta}) = e^{in\theta}$, n any integer, represent the different equivalence classes. The nth class consists of the characters $c_n(\alpha)\|\alpha\|^s$, which we denote by $c_n\|\ \|^s$.

The Corresponding Functions of $\mathfrak{z}$: We put

$$f_n(\xi) = \begin{cases} (x-iy)^{|n|}e^{-2\pi(x^2+y^2)}, & n \geq 0 \\ (x+iy)^{|n|}e^{-2\pi(x^2+y^2)}, & n \leq 0 \end{cases}$$

Their Fourier Transforms: We contend

$$\hat{f}_n(\xi) = i^{|n|} f_{-n}(\xi), \text{ for all } n.$$

Let us first establish this formula for $n \geq 0$ by induction. For $n = 0$, the contention is simply that $f_0(\xi) = e^{-2\pi(x^2+y^2)}$ is its own Fourier transform. This can be shown by breaking up the Fourier integral over the complex plane into a product of two reals and using again the classical formula

$$\int_{-\infty}^{+\infty} e^{-\pi u^2 + 2\pi i x u} du = e^{-\pi x^2}.$$

(The factor 2 in the exponent of our function $f_0(\xi)$ just compensates the factor 2 in $d\xi$ and in $\Lambda(\xi)$).

Assume now we have proved the contention for some $n \geq 0$. This means we have established the formula

$$\int f_n(\eta)\, e^{-2\pi i \Lambda(\xi\eta)} d\eta = i^n f_{-n}(\xi)$$

which, written out, becomes

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (u-iv)^n e^{-2\pi(u^2+v^2) + 4\pi i (xu - yv)}\, 2\,du\,dv = i^n (x+iy)^n e^{-2\pi(x^2+y^2)}.$$

Applying the operator $D = \frac{1}{4\pi i}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$ to both sides, (a simple task in view of the fact that since z^n is analytic, $D(x+iy)^n = 0$), we obtain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (u-iv)^{n+1} e^{-2\pi(u^2+v^2) + 4\pi i (xu - yv)}\, 2\,du\,dv = i^{n+1} (x+iy)^{n+1} e^{-2\pi(x^2+y^2)}.$$

This is the contention for $n+1$. The induction step is carried out.

To handle the case $n<0$, put a roof on the formula $\hat{f}_{-n}(\xi) = i^{|n|} f_n(\xi)$ which we have already proved, and remember that $\hat{\hat{f}}_{-n}(\xi) = f_{-n}(-\xi) = (-1)^{|n|} f_{-n}(\xi)$.

The ζ-Functions: For $\alpha = re^{i\theta}$ we have

$$f_n(\alpha) = r^{|n|} e^{-in\theta} e^{-2\pi r^2} \qquad |\alpha|^s = r^{2s}$$

$$c_n(\alpha) = e^{in\theta} \qquad d\alpha = \frac{2r\,dr\,d\theta}{r^2}.$$

Therefore

$$\zeta(f_n, c_n|\,|^s) = \int f_n(\alpha)c_n(\alpha)|\alpha|^s d^\times\alpha = \int_0^\infty\int_0^{2\pi} r^{2(s-1)+|n|}e^{-2\pi r^2}2r\,dr\,d\theta$$

$$= 2\pi\int_0^\infty (r^2)^{(s-1)+\frac{|n|}{2}}e^{-2\pi r^2}dr^2 = (2\pi)^{(1-s)+\frac{|n|}{2}}\Gamma\left(s+\frac{|n|}{2}\right)$$

and

$$\zeta(\hat{f_n}, \widehat{c_n|\,|^s}) = \zeta(i^{|n|}f_{-n}, c_{-n}|\,|^{1-s}) = i^{|n|}(2\pi)^{s+\frac{|n|}{2}}\Gamma\left(1-s+\frac{|n|}{2}\right)$$

Explicit Expressions for $\rho(c)$:

$$\rho(c_n|\,|^s) = (-i)^{|n|}\frac{(2\pi)^{1-s}\,\Gamma\left(s+\frac{|n|}{2}\right)}{(2\pi)^s\,\Gamma\left((1-s)+\frac{|n|}{2}\right)}$$

k $\mathfrak{p}$-adic.

ξ = a $\mathfrak{p}$-adic variable.

$\Lambda(\xi) = \lambda(s(\xi))$.

$d\xi$ is chosen so that $\mathfrak{o}$ gets measure $(N\mathfrak{d})^{-\frac{1}{2}}$

$\alpha = \tilde{\alpha}\pi^\nu$, non-zero $\mathfrak{p}$-adic variable, π a fixed element of ordinal number 1, ν an integer.

$|\alpha| = (N\mathfrak{p})^{-\nu}$

$d^\times\alpha = \dfrac{N\mathfrak{p}}{N\mathfrak{p}-1}\,\dfrac{d\alpha}{|\alpha|}$, so that u gets multiplicative measure $(N\mathfrak{d})^{-\frac{1}{2}}$.

The Equivalence Classes of Quasi-Characters: $c_n(\alpha)$, for $n \geq 0$ shall denote any character of $k^\times$ with conductor exactly $\mathfrak{p}^n$, such that $c_n(\pi)=1$. These characters represent the different equivalence classes of quasi-characters.

The Corresponding Functions of ζ: We put

$$f_n(\xi) = \begin{cases} e^{2\pi i\Lambda(\xi)} & \text{, for } \xi \in \mathfrak{d}^{-1}\mathfrak{p}^{-n} \\ 0 & \text{, for } \xi \notin \mathfrak{d}^{-1}\mathfrak{p}^{-n}. \end{cases}$$

Their Fourier Transforms: We contend

$$\hat{f}_n(\xi) = \begin{cases} (N\mathfrak{d})^{+\frac{1}{2}}(N\mathfrak{p})^n & \text{for } \xi \equiv 1 \pmod{\mathfrak{p}^n} \\ 0 & \text{for } \xi \not\equiv 1 \pmod{\mathfrak{p}^n} \end{cases}$$

Proof:

$$\hat{f}_n(\xi) = \int f_n(\eta) e^{-2\pi i \Lambda(\xi\eta)} d\eta = \int_{\mathfrak{d}^{-1}\mathfrak{p}^{-n}} e^{-2\pi i \Lambda((\xi-1)\eta)} d\eta$$

This is the integral, over the compact subgroup $\mathfrak{d}^{-1}\mathfrak{p}^{-n} \subset k^+$, of the additive character $\eta \rightarrow e^{-2\pi i \Lambda((\xi-1)\eta)}$. If $\xi \equiv 1 \pmod{\mathfrak{p}^n}$, this character is trivial on the subgroup, and the integral is simply the measure of the subgroup: $N\mathfrak{d}^{\frac{1}{2}} N\mathfrak{p}^n$. In case $\xi \not\equiv 1 \pmod{\mathfrak{p}^n}$, this character is not trivial on the subgroup and the integral is 0.

The ζ-Functions: First we treat the unramified case: $n = 0$. The only character of type c_0 is the identity character, and f_0 is the characteristic function of the set $\mathfrak{d}^{-1}$. We shall therefore compute

$$\zeta(f_0, |\ |^s) = \int_{\mathfrak{d}^{-1}} |\alpha|^s d^\times\alpha .$$

Denote by A_ν the "annulus" of elements of order ν, and let $\mathfrak{d} = \mathfrak{p}^d$. Then $\mathfrak{d}^{-1} = \bigcup_{\nu=-d}^{\infty} A_\nu$, a disjoint union, and

$$\begin{aligned}\zeta(f_0, |\ |^s) &= \sum_{\nu=-d}^{\infty} \int_{A_\nu} |\alpha|^s d^\times\alpha = \sum_{\nu=-d}^{\infty} N\mathfrak{p}^{-\nu s} \int_{A_\nu} d^\times\alpha \\ &= \left(\sum_{\nu=-d}^{\infty} N\mathfrak{p}^{-\nu s}\right) N\mathfrak{d}^{-\frac{1}{2}} = \frac{N\mathfrak{p}^{ds}}{1 - N\mathfrak{p}^{-s}} N\mathfrak{d}^{-\frac{1}{2}} \\ &= \frac{N\mathfrak{d}^{s-\frac{1}{2}}}{1 - N\mathfrak{p}^{-s}} ,\end{aligned}$$

$\hat{f}_0$ is $N\mathfrak{d}^{\frac{1}{2}}$ times the characteristic function of $\mathfrak{o}$, so we have, similarly,

$$\zeta(\hat{f}_0, \widehat{|\ |^s}) = \zeta(\hat{f}_0, |\ |^{1-s}) = N\mathfrak{d}^{\frac{1}{2}} \int_{\mathfrak{o}} |\alpha|^{1-s} d\alpha$$

$$= \sum_{\nu=0}^{\infty} N\mathfrak{p}^{-\nu(1-s)} = \frac{1}{1 - N\mathfrak{p}^{s-1}} .$$

In the ramified case, $n > 0$,

$$\zeta(f_n, c_n|\ |^s) = \int_{\mathfrak{d}^{-1}\mathfrak{p}^{-n}} e^{2\pi i \Lambda(\alpha)} c_n(\alpha)|\alpha|^s d\alpha$$

$$= \sum_{\nu=-d-n}^{\infty} N\mathfrak{p}^{-\nu s} \int_{A_\nu} e^{2\pi i \Lambda(\alpha)} c_n(\alpha)\, d\alpha$$

We assert that all terms in this sum after the first are 0. In other words that

$$\int_{A_\nu} e^{2\pi i \Lambda(\alpha)} c_n(\alpha)\, d\alpha = 0 \quad , \text{for } \nu > -d-n .$$

<u>Proof:</u> Case 1.) $\nu \geq -d$. Then $A_\nu \subset \mathfrak{d}^{-1}$, so $e^{2\pi i \Lambda(\alpha)} = 1$ on A_ν, and the integral is

$$\int_{A_\nu} c_n(\alpha) d\alpha = \int_{\mathfrak{u}} c_n(\alpha\pi^\nu)\, d\alpha = \int_{\mathfrak{u}} c_n(\alpha)\, d\alpha = 0.$$

since $c_n(\alpha)$ is ramified and therefore non-trivial on the subgroup $\mathfrak{u}$.

Case 2.) $-d > \nu > -d-n$ (Occurs only if there is "higher ramification"; i.e. if $n > 1$. To handle this case we break up A_ν into disjoint sets of the type $\alpha_0 + \mathfrak{d}^{-1} = \alpha_0 + \mathfrak{p}^{-d} = \alpha_0(1 + \mathfrak{p}^{-d-\nu})$. On such a set, Λ is constant $= \Lambda(\alpha_0)$ and

$$\int_{\alpha_0 + \mathfrak{d}^{-1}} e^{2\pi i \Lambda(\alpha)} c_n(\alpha) d\alpha = e^{2\pi i \Lambda(\alpha_0)} \int_{\alpha_0 + \mathfrak{d}^{-1}} c_n(\alpha)\, d\alpha$$

This is 0 because

$$\int_{\alpha_0 + \mathfrak{d}^{-1}} c_n(\alpha)\, d\alpha = \int_{\alpha_0(1+\mathfrak{p}^{-d-\nu})} c_n(\alpha) d\alpha = \int_{1+\mathfrak{p}^{-d-\nu}} c_n(\alpha\alpha_0)\, d\alpha = c_n(\alpha_0) \int_{1+\mathfrak{p}^{-d-\nu}} c_n(\alpha)\, d\alpha ,$$

and this last integral is the integral over a multiplicative subgroup $1 + \mathfrak{p}^{-d-\nu}$ of a character $c_n(\alpha)$ which is not trivial on the subgroup. Namely, $-d > \nu \Rightarrow \mathfrak{p} \mid \mathfrak{p}^{-d-\nu} \Rightarrow 1 + \mathfrak{p}^{-d-\nu}$ is a subgroup of $k^\times$, and $\nu > -d-n \Rightarrow$ the conductor $\mathfrak{p}^n \nmid \mathfrak{p}^{-d-\nu} \Rightarrow c_n(\alpha)$ not trivial on it.

We have now shown

$$\zeta(f_n, c_n||^s) = N\mathfrak{p}^{(d+n)s} \int_{A_{-d-n}} e^{2\pi i \Lambda(\alpha)} c_n(\alpha)\, d\alpha .$$

To write this in a better ~~~~ form, let $\{\varepsilon\}$ be a set of representatives of the elements of the factor group $\mathfrak{u}/1+\mathfrak{p}^n$, so that $\mathfrak{u} = \bigcup_\varepsilon \varepsilon(1+\mathfrak{p}^n)$, a disjoint union. Then

$$A_{-d-n} = \mathfrak{u}\pi^{-d-n} = \bigcup_\varepsilon \varepsilon\pi^{-d-n}(1+\mathfrak{p}^n) = \bigcup_\varepsilon (\varepsilon\pi^{-d-n} + \mathfrak{d}^{-1}).$$

On each of these sets into which we have dissected A_{-d-n}, c_n is constant $= c_n(\varepsilon\pi^{-d-n}) = c_n(\varepsilon)$, and Λ is constant $= \Lambda(\varepsilon\pi^{-d-n})$. We therefore have

$$\zeta(f_n, c_n||^s) = N\mathfrak{p}^{(d+n)s}\left(\sum_\varepsilon c_n(\varepsilon) e^{2\pi i \Lambda(\frac{\varepsilon}{\pi^{d+n}})}\right) \int_{1+\mathfrak{p}^n} d\alpha ,$$

a form which will be convenient enough.

The pay-off comes in computing $\zeta(\widehat{f_n}, \widehat{c_n||^s}) = \zeta(\widehat{f_n}, c_n^{-1}||^{1-s})$. For $\widehat{f_n}$ is $N\mathfrak{d}^{-\frac{1}{2}} N\mathfrak{p}^n$ times the characteristic function of the set $1+\mathfrak{p}^n$, a set on which $c_n^{-1}(\alpha)|\alpha|^{1-s} = 1$. Therefore

$$\zeta(\widehat{f_n}, \widehat{c_n||^s}) = N\mathfrak{d}^{\frac{1}{2}} N\mathfrak{p}^n \int_{1+\mathfrak{p}^n} d\alpha \quad , \text{ a constant!}$$

<u>Explicit Expressions for $\rho(c)$:</u>

$$\rho(||^s) = N\mathfrak{d}^{s-\frac{1}{2}} \frac{1 - N\mathfrak{p}^{s-1}}{1 - N\mathfrak{p}^{-s}} .$$

$\rho(c||^s) = N(\mathfrak{d}\mathfrak{f})^{s-\frac{1}{2}} \rho_0(c)$, if c is a ramified character with conductor $\mathfrak{f}$, such that $c(\pi) = 1$. $\rho_0(c) = N\mathfrak{f}^{-\frac{1}{2}} \sum_\varepsilon c(\varepsilon) e^{2\pi i \Lambda(\frac{\varepsilon}{\pi^{\mathrm{ord}\,\mathfrak{d}\mathfrak{f}}})}$ is a so-called root number and has absolute value 1. $\{\varepsilon\}$ is a set of representatives of the cosets of $1+\mathfrak{f}$ in $\mathfrak{u}$.

Taking the quotients of the ζ-functions we have worked out yields these expressions directly if we remember that $\mathfrak{d} = \mathfrak{p}^d$ and, in the ramified case, that the conductor of c_n was $\mathfrak{f} = \mathfrak{p}^n$. The fact that the constant $\rho_0(c)$ has absolute value 1 follows from corollary 2.4.1.

Namely, since c is a character, $c\,|\,|^{\frac{1}{2}}$ has exponent $\frac{1}{2}$, so we must have $|\rho(c\,|\,|^{\frac{1}{2}})| = |\rho_0(c)| = 1.$

Chapter 3.

Abstract Restricted Direct Product.

3.1 Introduction. Let $\{\mathfrak{p}\}$ be a set of indices. Suppose we are given for each $\mathfrak{p}$ a locally compact abelian group $G_{\mathfrak{p}}$, and for almost all $\mathfrak{p}$ (meaning for all but a finite number of $\mathfrak{p}$), a fixed subgroup $H_{\mathfrak{p}} \subset G_{\mathfrak{p}}$ which is open and compact.

We may then form a new abstract group G whose elements $\mathfrak{a} = (\ldots\ldots\ldots, \mathfrak{a}_{\mathfrak{p}}, \ldots\ldots\ldots)$ are "vectors" having one component $\mathfrak{a}_{\mathfrak{p}} \in G_{\mathfrak{p}}$ for each $\mathfrak{p}$, with $\mathfrak{a}_{\mathfrak{p}} \in H_{\mathfrak{p}}$ for almost all $\mathfrak{p}$. Multiplication is defined component wise.

Let S be a finite set of indices $\mathfrak{p}$, including at least all those $\mathfrak{p}$ for which $H_{\mathfrak{p}}$ is not defined. The elements $\mathfrak{a} \in G$ such that $\mathfrak{a}_{\mathfrak{p}} \in H_{\mathfrak{p}}$ for $\mathfrak{p} \notin S$ comprise a subgroup of G which we denote by G_S. G_S is naturally isomorphic to a direct product $\prod_{\mathfrak{p} \in S} G_{\mathfrak{p}} \times \prod_{\mathfrak{p} \notin S} H_{\mathfrak{p}}$ of locally compact groups, almost all of which are compact, and is therefore a locally compact group in the product topology. We define a topology in G by taking as a fundamental system of neighborhoods of 1 in G, the set of neighborhoods of 1 in G_S. The resulting topology in G does not depend on the set of indices, S, which we selected. This can be seen from

Lemma 3.1.1 The totality of all "parallelotopes" of the form $N = \prod_{\mathfrak{p}} N_{\mathfrak{p}}$, where $N_{\mathfrak{p}}$ is a neighborhood of 1 in $G_{\mathfrak{p}}$ for all $\mathfrak{p}$, and $N_{\mathfrak{p}} = H_{\mathfrak{p}}$ for almost all $\mathfrak{p}$ — remember the $H_{\mathfrak{p}}$ are open by hypothesis — is a fundamental system of neighborhoods of 1 in G.

Proof: By the definition of product topology a neighborhood of 1 in G_S contains a parallelotope of the type described. On the other hand, since $N_{\mathfrak{p}} = H_{\mathfrak{p}}$ for almost all $\mathfrak{p}$, the intersection $(\prod_{\mathfrak{p}} N_{\mathfrak{p}}) \cap G_S = \prod_{\mathfrak{p} \in S} N_{\mathfrak{p}} \times \prod_{\mathfrak{p} \notin S} (N_{\mathfrak{p}} \cap H_{\mathfrak{p}})$ is a neighborhood of 1 in G_S.

It is obvious that G_S is open in G and that the topology induced in G_S as a subspace of G is the same as the product topology we imposed on G_S to begin with. Therefore a compact neighborhood of 1 in G_S is a compact neighborhood of 1 in G. It follows that G is locally compact.

Definition 3.1.1: We call G (as locally compact abelian group) the restricted direct product of the groups $G_{\mathfrak{p}}$ relative to the subgroups $H_{\mathfrak{p}}$.

It will, of course, be convenient to identify the basic group $G_{\mathfrak{p}}$ with the subgroup of G consisting of the elements $\mathfrak{M}_{\mathfrak{p}} = (1, 1, \ldots, \mathfrak{M}_{\mathfrak{p}}, \ldots)$ having all components but the $\mathfrak{p}^{th}$ equal to 1. For that subgroup of G is naturally isomorphic, both topologically and algebraically to $G_{\mathfrak{p}}$.

Since the components, $\mathfrak{M}_{\mathfrak{p}}$, of any element $\mathfrak{M}$ of G lie in $H_{\mathfrak{p}}$ for almost all $\mathfrak{p}$, G is the union of the subgroups of the type G_S. This fact will allow us to reduce our investigations of G to a study of the subgroups G_S.

These G_S in turn may be effectively analysed by introducing the subgroup $G^S \subset G_S$ consisting of all elements $\mathfrak{M} \in G$ such that $\mathfrak{M}_{\mathfrak{p}} = 1$ for $\mathfrak{p} \in S$; $\mathfrak{M}_{\mathfrak{p}} \in H_{\mathfrak{p}}$, $\mathfrak{p} \notin S$. G^S is compact since it is naturally isomorphic to a direct product $\prod_{\mathfrak{p} \notin S} H_{\mathfrak{p}}$ of compact groups. G_S can be considered as the direct product $G_S = (\prod_{\mathfrak{p} \in S} G_{\mathfrak{p}}) \times G^S$ of a finite number of our basic groups $G_{\mathfrak{p}}$ and the compact group G^S.

We close our introduction of the restricted direct product with

Lemma 3.1.2: A subset $C \subset G$ is relatively compact (has a compact closure) if, and only, if it is contained in a parallelotope of the type $\prod_{\mathfrak{p}} B_{\mathfrak{p}}$, where $B_{\mathfrak{p}}$ is a compact subset of $G_{\mathfrak{p}}$ for all $\mathfrak{p}$, and $B_{\mathfrak{p}} = H_{\mathfrak{p}}$ for almost all $\mathfrak{p}$.

Proof: Any compact subset of G is contained in some G_S, because the G_S are open sets covering G, and the union of a finite number of subgroups G_S is again a G_S. Any compact subset of a G_S is contained in a parallelotope of the type described, for it is contained in the cartesian product of its "projections" onto the component groups $G_{\mathfrak{p}}$. These projections are compact since they are continuous images, and are contained in $H_{\mathfrak{p}}$ for $\mathfrak{p} \notin S$.

On the other hand, any parallelotope $\prod_{\mathfrak{p}} B_{\mathfrak{p}}$ is obviously a compact subset of some G_S; therefore of G.

3.2. Characters. Let $c(\mathfrak{a})$ be a quasi-character of G, i.e. a continuous multiplicative mapping of G into the complex numbers. We denote by $c_{\mathfrak{p}}$ the restriction of c to $G_{\mathfrak{p}}$: ($c_{\mathfrak{p}}(\mathfrak{a}_{\mathfrak{p}}) = c(\mathfrak{a}_{\mathfrak{p}}) = c(1, 1, \ldots, \mathfrak{a}_{\mathfrak{p}}, \ldots)$ for $\mathfrak{a}_{\mathfrak{p}} \in G_{\mathfrak{p}}$). $c_{\mathfrak{p}}$ is obviously a quasi-character of $G_{\mathfrak{p}}$.

Lemma 3.2.1: $c_{\mathfrak{p}}$ is trivial on $H_{\mathfrak{p}}$, for almost all $\mathfrak{p}$, and we have for any $\mathfrak{a} \in G$

$$c(\mathfrak{a}) = \prod_{\mathfrak{p}} c_{\mathfrak{p}}(\mathfrak{a}_{\mathfrak{p}})$$

almost all factors of the product being 1.

Proof: Let U be a neighborhood of 1 in the complex numbers containing no multiplicative subgroup except $\{1\}$. Let $N = \prod_{\mathfrak{p}} N_{\mathfrak{p}}$ be a neighborhood of 1 in G such that $c(N) \subset U$. Select an S containing all $\mathfrak{p}$ for which $N_{\mathfrak{p}} \neq H_{\mathfrak{p}}$. Then $G^S \subset N \Rightarrow c(G^S) \subset U \Rightarrow c(G^S) = 1 \Rightarrow c(H_{\mathfrak{p}}) = 1$ for $\mathfrak{p} \notin S$. If $\mathfrak{a}$ is a fixed element of G we impose on S the further condition that $\mathfrak{a} \in G_S$ and write $\mathfrak{a} = \left(\prod_{\mathfrak{p} \in S} \mathfrak{a}_{\mathfrak{p}}\right) \mathfrak{a}^S$ with $\mathfrak{a}^S \in G^S$. Then

$$c(\mathfrak{a}) = \prod_{\mathfrak{p} \in S} c(\mathfrak{a}_{\mathfrak{p}}) \cdot c(\mathfrak{a}^S) = \prod_{\mathfrak{p} \in S} c_{\mathfrak{p}}(\mathfrak{a}_{\mathfrak{p}}) = \prod_{\mathfrak{p}} c_{\mathfrak{p}}(\mathfrak{a}_{\mathfrak{p}}),$$

since for $\mathfrak{p} \notin S$, $c_{\mathfrak{p}}(\mathfrak{a}_{\mathfrak{p}}) = 1$.

Lemma 3.2.2: Let $c_{\mathfrak{p}}$ be a given quasi-character of $G_{\mathfrak{p}}$ for each $\mathfrak{p}$, with $c_{\mathfrak{p}}$ trivial on $H_{\mathfrak{p}}$ for almost all $\mathfrak{p}$. Then if we define $c(\mathfrak{a}) = \prod_{\mathfrak{p}} c_{\mathfrak{p}}(\mathfrak{a}_{\mathfrak{p}})$

we obtain a quasi-character of G.

<u>Proof:</u> $c(\mathfrak{a})$ is obviously multiplicative. To see that it is continuous select an S containing all $\mathfrak{p}$ for which $c_{\mathfrak{p}}(H_{\mathfrak{p}}) \neq 1$. Let s be the number of $\mathfrak{p}$ in S. Given a neighborhood, U, of 1 in the complex numbers, choose a neighborhood V such that $V^s \subset U$. Let $N_{\mathfrak{p}}$ be a neighborhood of 1 in $G_{\mathfrak{p}}$ such that $c_{\mathfrak{p}}(N_{\mathfrak{p}}) \subset V$ for $\mathfrak{p} \in S$, and let $N_{\mathfrak{p}} = H_{\mathfrak{p}}$ for $\mathfrak{p} \notin S$. Then $c(\prod_{\mathfrak{p}} N_{\mathfrak{p}}) \subset V^s \subset U$.

Restricting our consideration to characters, we notice first of all that $c(\mathfrak{a}) = \prod_{\mathfrak{p}} c_{\mathfrak{p}}(\mathfrak{a}_{\mathfrak{p}})$ is a character if, and only if all $c_{\mathfrak{p}}$ are characters. Denote by $\hat{G}_{\mathfrak{p}}$ the character group of $G_{\mathfrak{p}}$, for all $\mathfrak{p}$; for the $\mathfrak{p}$ where $H_{\mathfrak{p}}$ is defined let $H_{\mathfrak{p}}^* \subset \hat{G}_{\mathfrak{p}}$ be the subgroup of all $c_{\mathfrak{p}} \in \hat{G}_{\mathfrak{p}}$ which are trivial on $H_{\mathfrak{p}}$. Then $H_{\mathfrak{p}}$ compact $\Rightarrow \hat{H}_{\mathfrak{p}} \cong \hat{G}_{\mathfrak{p}}/H_{\mathfrak{p}}^*$ discrete $\Rightarrow H_{\mathfrak{p}}^*$ open, and $H_{\mathfrak{p}}$ open $\Rightarrow G_{\mathfrak{p}}/H_{\mathfrak{p}}$ discrete $\Rightarrow \widehat{G_{\mathfrak{p}}/H_{\mathfrak{p}}} \cong H_{\mathfrak{p}}^*$ compact.

<u>Theorem 3.2.1:</u> The restricted direct product of the groups $\hat{G}_{\mathfrak{p}}$ relative to the subgroups $H_{\mathfrak{p}}^*$ is naturally isomorphic, both topologically and algebraically, to the character group $\hat{G}$ of G.

<u>Proof:</u> Of course we mean to identify $c = (\dots, c_{\mathfrak{p}}, \dots)$ with the character $c(\mathfrak{a}) = \prod_{\mathfrak{p}} c_{\mathfrak{p}}(\mathfrak{a}_{\mathfrak{p}})$. The two preceding lemmas, applied to characters, show that this is an algebraic isomorphism between the two groups. We have only to check that the topology is the same. To this effect we reason as follows: $c = (\dots, c_{\mathfrak{p}}, \dots)$ is close to 1 as a character $\Longleftrightarrow$ $c(B)$ close to 1 for a large compact $B \subset G$ $\Longleftrightarrow$ $c(\prod B_{\mathfrak{p}})$ close to 1 for $B_{\mathfrak{p}} \subset G_{\mathfrak{p}}$, compact, $B_{\mathfrak{p}} = H_{\mathfrak{p}}$ for almost all $\mathfrak{p}$ $\Longleftrightarrow$ $c_{\mathfrak{p}}(B_{\mathfrak{p}})$ close to 1 wherever $B_{\mathfrak{p}} \neq H_{\mathfrak{p}}$ and $c_{\mathfrak{p}}(B_{\mathfrak{p}}) = c_{\mathfrak{p}}(H_{\mathfrak{p}}) = 1$ at the remaining $\mathfrak{p}$ (since $H_{\mathfrak{p}}$ is a subgroup, $c_{\mathfrak{p}}(H_{\mathfrak{p}})$ can be close to 1 only if $c_{\mathfrak{p}}(H_{\mathfrak{p}}) = 1$)

$\Leftrightarrow$ $c_{\mathfrak{g}}$ close to 1 in $\hat{G}_{\mathfrak{g}}$ for a finite number of $\mathfrak{g}$ and $c_{\mathfrak{g}} \in H^*_{\mathfrak{g}}$ at the other $\mathfrak{g}$ $\Leftrightarrow$ c close to 1 in the restricted direct product of the $\hat{G}_{\mathfrak{g}}$.

3.3 Measure. Assume now that we have chosen a Haar measure $d\mathfrak{M}_{\mathfrak{g}}$ on each $G_{\mathfrak{g}}$ such that $\int_{H_{\mathfrak{g}}} d\mathfrak{M}_{\mathfrak{g}} = 1$ for almost all $\mathfrak{g}$. We wish to define a Haar measure $d\mathfrak{M}$ on G for which, in some sense, $d\mathfrak{M} = \prod_{\mathfrak{g}} d\mathfrak{M}_{\mathfrak{g}}$. To do this, we select an S; then consider G_S as the finite direct product $G_S = (\prod_{\mathfrak{g} \in S} G_{\mathfrak{g}}) \times G^S$, in order to define on G_S a measure $d\mathfrak{M}_S = (\prod_{\mathfrak{g} \in S} d\mathfrak{M}_{\mathfrak{g}}) \cdot d\mathfrak{M}^S$, where $d\mathfrak{M}^S$ is that measure on the compact group G^S for which $\int_{G^S} d\mathfrak{M}^S = \prod_{\mathfrak{g} \notin S} \left[\int_{H_{\mathfrak{g}}} d\mathfrak{M}_{\mathfrak{g}} \right]$. Since G_S is an open subgroup of G, a Haar measure $d\mathfrak{M}$ on G is now determined by the requirement that $d\mathfrak{M} = d\mathfrak{M}_S$ on G_S. To see that the $d\mathfrak{M}$ we have just chosen is really independent of the set S, let $T \supset S$ be a larger set of indices. Then $G_S \subset G_T$, and we have only to check that the $d\mathfrak{M}_T$ constructed with T coincides on G_S with the $d\mathfrak{M}_S$ constructed with S. Now one sees from the decomposition $G^S = (\prod_{\mathfrak{g} \in T-S} H_{\mathfrak{g}}) \times G^T$ that $d\mathfrak{M}^S = (\prod_{\mathfrak{g} \in T-S} d\mathfrak{M}_{\mathfrak{g}}) \cdot d\mathfrak{M}^T$; for the measure on the righthand side gives to the compact group G^S the required measure. Therefore

$$d\mathfrak{M}_S = \prod_{\mathfrak{g} \in S} d\mathfrak{M}_{\mathfrak{g}} \cdot d\mathfrak{M}^S = \prod_{\mathfrak{g} \in S} d\mathfrak{M}_{\mathfrak{g}} \prod_{\mathfrak{g} \in T-S} d\mathfrak{M}_{\mathfrak{g}} \cdot d\mathfrak{M}^T = d\mathfrak{M}_T .$$

We have therefore determined a unique Haar measure $d\mathfrak{M}$ on G which we may denote symbolically by $d\mathfrak{M} = \prod_{\mathfrak{g}} d\mathfrak{M}_{\mathfrak{g}}$.

If $\varphi(S)$ is any function of the finite sets of indices S, with values in a topological space, we shall mean by the expression $\lim_S \varphi(S) = \varphi_0$ the statement: "given any neighborhood V of φ_0, there exists a set $S(V)$ such that $S \supset S(V) \Rightarrow \varphi(S) \in V$". Intuitively, $\lim_S \varphi(S)$ means the limit of $\varphi(S)$ as S becomes larger and larger.

Lemma 3.3.1: If $f(\mathfrak{M})$ is a function on G,

$$\int f(\mathfrak{M})\, d\mathfrak{M} = \lim_S \int_{G_S} f(\mathfrak{M})\, d\mathfrak{M} ,$$

if either 1.) $f(\mathfrak{M})$ measurable, $f(\mathfrak{M}) \geq 0$, in which case $+\infty$ is allowed as value of the integrals; or 2.) $f(\mathfrak{M}) \in L_1(G)$, in which case the values of the integrals are complex numbers.

<u>Proof:</u> In either case 1.) or 2.) $\int f(\mathfrak{M})d\mathfrak{M}$ is the limit of $\int_B f(\mathfrak{M})d\mathfrak{M}$ for larger and larger compacts $B \subset G$. Since any compact, B, is contained in some G_S, the statement follows.

<u>Lemma 3.3.2:</u> Assume we are given for each $\mathfrak{y}$ a continuous function $f_{\mathfrak{y}} \in L_1(G_{\mathfrak{y}})$ such that $f_{\mathfrak{y}}(\mathfrak{M}_{\mathfrak{y}}) = 1$ on $H_{\mathfrak{y}}$ for almost all $\mathfrak{y}$. We define on G the function $f(\mathfrak{M}) = \prod_{\mathfrak{y}} f_{\mathfrak{y}}(\mathfrak{M}_{\mathfrak{y}})$, (this is really a finite product), and contend:

1.) $f(\mathfrak{M})$ is continuous on G.

2.) For any set S containing at least those $\mathfrak{y}$ for which either $f_{\mathfrak{y}}(H_{\mathfrak{y}}) \neq 1$, or $\int_{H_{\mathfrak{y}}} d\mathfrak{M}_{\mathfrak{y}} \neq 1$, we have

$$\int_{G_S} f(\mathfrak{M})d\mathfrak{M} = \prod_{\mathfrak{y}\in S}\left[\int_{G_{\mathfrak{y}}} f_{\mathfrak{y}}(\mathfrak{M}_{\mathfrak{y}})\,d\mathfrak{M}_{\mathfrak{y}}\right].$$

<u>Proof:</u> 1.) $f(\mathfrak{M})$ is obviously continuous on any G_S; therefore on G.

2.) For $\mathfrak{M} \in G_S$, $f(\mathfrak{M}) = \prod_{\mathfrak{y}\in S} f_{\mathfrak{y}}(\mathfrak{M}_{\mathfrak{y}})$. Hence

$$\int_{G_S} f(\mathfrak{M})d\mathfrak{M} = \int_{G_S} f(\mathfrak{M})d\mathfrak{M}_S = \int_{G_S}\Big(\prod_{\mathfrak{y}\in S} f_{\mathfrak{y}}(\mathfrak{M}_{\mathfrak{y}})\Big)\Big(\prod_{\mathfrak{y}\in S} d\mathfrak{M}_{\mathfrak{y}}\cdot d\mathfrak{M}^S\Big)$$

$$= \prod_{\mathfrak{y}\in S}\left[\int f_{\mathfrak{y}}(\mathfrak{M}_{\mathfrak{y}})\,d\mathfrak{M}_{\mathfrak{y}}\right]\cdot\int_{G^S} d\mathfrak{M}^S = \prod_{\mathfrak{y}\in S}\left[\int f_{\mathfrak{y}}(\mathfrak{M}_{\mathfrak{y}})\,d\mathfrak{M}_{\mathfrak{y}}\right].$$

<u>Theorem 3.3.1:</u> If $f_{\mathfrak{y}}(\mathfrak{M}_{\mathfrak{y}})$ and $f(\mathfrak{M})$ are the functions of the preceding lemma and if furthermore

$$\prod_{\mathfrak{y}}\left[\int |f_{\mathfrak{y}}(\mathfrak{M}_{\mathfrak{y}})|\,d\mathfrak{M}_{\mathfrak{y}}\right] \quad \Big(= \lim_{S}\Big\{\prod_{\mathfrak{y}\in S}\Big[\int |f_{\mathfrak{y}}(\mathfrak{M}_{\mathfrak{y}})|\,d\mathfrak{M}_{\mathfrak{y}}\Big]\Big\}\Big) < \infty$$

then $f(\mathfrak{M}) \in L_1(G)$ and,

$$\int f(\mathfrak{M})\,d\mathfrak{M} = \prod_{\mathfrak{y}}\left[\int f_{\mathfrak{y}}(\mathfrak{M}_{\mathfrak{y}})\,d\mathfrak{M}_{\mathfrak{y}}\right].$$

Proof: Combine the two preceding lemmas; first for the function $|f(\mathfrak{M})| = \prod_{\mathfrak{y}} |f_{\mathfrak{y}}(\mathfrak{M}_{\mathfrak{y}})|$ to see that $f(\mathfrak{M}) \in L_1(G)$, then for $f(\mathfrak{M})$ itself to evaluate $\int f(\mathfrak{M})d\mathfrak{M}$.

We close this chapter with some remarks about Fourier analysis in a restricted direct product. As we have seen, $\hat{G}$ the character group of G is the direct product of the character groups $\hat{G}_{\mathfrak{y}}$ of $G_{\mathfrak{y}}$, relative to the subgroups $H^*_{\mathfrak{y}}$ orthogonal to $H_{\mathfrak{y}}$. Denote by $c = (\dots, c_{\mathfrak{y}}, \dots)$ the general element of $\hat{G}$. (In this paragraph, c, and $c_{\mathfrak{y}}$ are characters, not quasi-characters). Let $dc_{\mathfrak{y}}$ be the measure in $\hat{G}_{\mathfrak{y}}$ dual to the measured $d\mathfrak{M}_{\mathfrak{y}}$ in $G_{\mathfrak{y}}$. Notice that if $f_{\mathfrak{y}}(\mathfrak{M}_{\mathfrak{y}})$ is the characteristic function of $H_{\mathfrak{y}}$, its Fourier transform $\hat{f}_{\mathfrak{y}}(c_{\mathfrak{y}}) = \int f_{\mathfrak{y}}(\mathfrak{M}_{\mathfrak{y}})\overline{c_{\mathfrak{y}}(\mathfrak{M}_{\mathfrak{y}})}d\mathfrak{M}_{\mathfrak{y}}$ is $\int_{H_{\mathfrak{y}}} d\mathfrak{M}_{\mathfrak{y}}$ times the characteristic function of $H^*_{\mathfrak{y}}$. A consequence of this fact and the inversion formula is that $\left(\int_{H_{\mathfrak{y}}} d\mathfrak{M}_{\mathfrak{y}}\right)\left(\int_{H^*_{\mathfrak{y}}} dc_{\mathfrak{y}}\right) = 1$. Therefore $\int_{H^*_{\mathfrak{y}}} dc_{\mathfrak{y}} = 1$ for almost all $\mathfrak{y}$, and we may put $dc = \prod_{\mathfrak{y}} dc_{\mathfrak{y}}$.

__Lemma 3.3.3:__ If $f_{\mathfrak{y}}(\mathfrak{M}_{\mathfrak{y}}) \in \mathcal{W}_1(G_{\mathfrak{y}})$ for all $\mathfrak{y}$ and $f_{\mathfrak{y}}(\mathfrak{M}_{\mathfrak{y}})$ is the characteristic function of $H_{\mathfrak{y}}$ for almost all $\mathfrak{y}$, then the function $f(\mathfrak{M}) = \prod_{\mathfrak{y}} f_{\mathfrak{y}}(\mathfrak{M}_{\mathfrak{y}})$ has the Fourier transform $\hat{f}(c) = \prod_{\mathfrak{y}} \hat{f}_{\mathfrak{y}}(c_{\mathfrak{y}})$, and $f(\mathfrak{M}) \in \mathcal{W}_1(G)$.

__Proof:__ Apply theorem 3.3.1 to the function $f(\mathfrak{M})\overline{c(\mathfrak{M})} = \prod_{\mathfrak{y}} f_{\mathfrak{y}}(\mathfrak{M}_{\mathfrak{y}})\overline{c_{\mathfrak{y}}(\mathfrak{M}_{\mathfrak{y}})}$ to see that the Fourier transform of the product is the product of the Fourier transforms. Since $f_{\mathfrak{y}}(\mathfrak{M}_{\mathfrak{y}}) \in \mathcal{W}_1(G_{\mathfrak{y}})$, $\hat{f}_{\mathfrak{y}}(c_{\mathfrak{y}}) \in L_1(\hat{G}_{\mathfrak{y}})$ for all $\mathfrak{y}$. For almost all $\mathfrak{y}$, $\hat{f}_{\mathfrak{y}}(c_{\mathfrak{y}})$ is the characteristic function of $H^*_{\mathfrak{y}}$ according to the remark above. From this we see that $\hat{f}(c) \in L_1(\hat{G})$, hence $f(\mathfrak{M}) \in \mathcal{W}_1(G)$.

__Corollary 3.3.1:__ The measure $dc = \prod_{\mathfrak{y}} dc_{\mathfrak{y}}$ is dual to $d\mathfrak{M} = \prod_{\mathfrak{y}} d\mathfrak{M}_{\mathfrak{y}}$.

__Proof:__ Applying the preceding lemma to the group $\hat{G}$ with the measure dc, we obtain for our "product" functions the inversion formula

$$f(\mathfrak{M}) = \int \hat{f}(c)c(\mathfrak{M})dc$$

from the component wise inversion formulas.

Chapter IV.

The Theory in the Large.

4.1. Additive Theory. In this chapter, k denotes a finite algebraic number field. $\mathfrak{p}$ is the generic prime divisor of k. The completion of k at the prime divisor $\mathfrak{p}$ shall from now on be denoted by $k_{\mathfrak{p}}$, and all the symbols $\mathfrak{o}, \Lambda, \vartheta, ||, c, \ldots$ etc. defined in Chapter II for this local field $k_{\mathfrak{p}}$ shall also receive the subscript $\mathfrak{p}$; $\mathfrak{o}_{\mathfrak{p}}, \Lambda_{\mathfrak{p}}, \vartheta_{\mathfrak{p}}, \ldots$ etc.

Definition 4.1.1: The additive group V of valuation vectors of k is the restricted direct sum, over all prime divisors $\mathfrak{p}$, of the groups $k_{\mathfrak{p}}^{+}$ relative to the subgroups $\mathfrak{o}_{\mathfrak{p}}$.

We shall denote the generic element of V (= valuation vector) by $\mathfrak{x} = (\ldots, \mathfrak{x}_{\mathfrak{p}}, \ldots)$. From theorems 3.2.1 and 2.2.1 and lemma 2.2.3 we see that the character group of V is naturally the restricted direct sum of the groups $k_{\mathfrak{p}}^{+}$ relative to the subgroups $\vartheta_{\mathfrak{p}}^{-1}$. Since $\vartheta_{\mathfrak{p}} = \mathfrak{o}_{\mathfrak{p}}$ for almost all $\mathfrak{p}$ this sum is simply V again! Looking more closely at the identifications set up in these theorems we see that the element $\mathfrak{y} = (\ldots, \mathfrak{y}_{\mathfrak{p}}, \ldots) \in V$ is to be identified with the character

$$\mathfrak{x} = (\ldots, \mathfrak{x}_{\mathfrak{p}}, \ldots) \to \prod_{\mathfrak{p}} e^{2\pi i \Lambda_{\mathfrak{p}}(\mathfrak{y}_{\mathfrak{p}} \mathfrak{x}_{\mathfrak{p}})} = e^{2\pi i \sum_{\mathfrak{p}} \Lambda_{\mathfrak{p}}(\mathfrak{y}_{\mathfrak{p}} \mathfrak{x}_{\mathfrak{p}})}$$

of V. This suggests that we define the additive function $\Lambda(\mathfrak{x}) = \sum_{\mathfrak{p}} \Lambda_{\mathfrak{p}}(\mathfrak{x}_{\mathfrak{p}})$ on V, and introduce component-wise multiplication $\mathfrak{y}\mathfrak{x} = (\ldots, \mathfrak{y}_{\mathfrak{p}}, \ldots)(\ldots, \mathfrak{x}_{\mathfrak{p}}, \ldots) = (\ldots, \mathfrak{y}_{\mathfrak{p}} \mathfrak{x}_{\mathfrak{p}}, \ldots)$ of elements of V in order to be able to assert neatly:

Theorem 4.1.1: V is naturally its own character group if we identify the element $\mathfrak{y} \in V$ with the character $\mathfrak{x} \to e^{2\pi i \Lambda(\mathfrak{y}\mathfrak{x})}$ of V.

On V we shall, of course, take the measure $d\mathfrak{x} = \prod_{\mathfrak{p}} d\mathfrak{x}_{\mathfrak{p}}$ described in § 3.3, $d\mathfrak{x}_{\mathfrak{p}}$ being the local additive measure defined in § 2.2. Since these local measures $d\mathfrak{x}_{\mathfrak{p}}$ were chosen to be self-dual, the same is true of $d\mathfrak{x}$, according to corollary 3.3.1. We state this fact formally in

<u>Theorem 3.1.2</u>: If for a function $f(\mathfrak{x}) \in L_1(V)$ we define the Fourier transform

$$\hat{f}(\mathfrak{y}) = \int f(\mathfrak{x}) e^{-2\pi i \Lambda(\mathfrak{y}\mathfrak{x})} d\mathfrak{x},$$

then for $f(\mathfrak{x}) \in \mathfrak{B}_1(V)$ the inversion formula

$$f(\mathfrak{x}) = \int \hat{f}(\mathfrak{y}) e^{2\pi i \Lambda(\mathfrak{x}\mathfrak{y})} d\mathfrak{y}$$

holds.

What is the analogue in the large of the local lemmas 2.2.4 and 2.2.5, that is, of the statement $d(\alpha\xi) = |\alpha| d\xi$ for $\alpha \in k_{\mathfrak{p}}^{\times}$? In that local consideration, α played the role of an auto-morphism of $k_{\mathfrak{p}}^{+}$, namely the automorphism $\xi \to \alpha\xi$. This leads us to investigate the question: for what $\mathfrak{a} \in V$ is $\mathfrak{x} \to \mathfrak{a}\mathfrak{x}$ an automorphism of V? We first observe that for any $\mathfrak{a} \in V$, $\mathfrak{x} \to \mathfrak{a}\mathfrak{x}$ is a continuous homomorphism of V into V. A necessary condition for it to be an automorphism is the existence of a $\mathfrak{b} \in V$ such that $\mathfrak{a}\mathfrak{b} = 1 = (1, 1, \ldots)$. But this is also sufficient, for with this $\mathfrak{b}$ we obtain an inverse map $\mathfrak{x} \to \mathfrak{b}\mathfrak{x}$ of the same form. Now for such a $\mathfrak{b}$ to exist at all as an "unrestricted" vector, we need $\mathfrak{a}_{\mathfrak{p}} \neq 0$ for all $\mathfrak{p}$, and then $\mathfrak{b}_{\mathfrak{p}} = \mathfrak{a}_{\mathfrak{p}}^{-1}$. The further condition $\mathfrak{b} \in V$ means $\mathfrak{a}_{\mathfrak{p}}^{-1} \in \mathfrak{o}_{\mathfrak{p}}$ for almost all $\mathfrak{p}$, therefore $|\mathfrak{a}_{\mathfrak{p}}|_{\mathfrak{p}} = 1$ for almost all $\mathfrak{p}$. These two conditions mean simply that $\mathfrak{a}$ is an idèle in the sense of Chevalley. We have proved

<u>Lemma 4.1.1</u>: The map $\mathfrak{a} \to \mathfrak{a}\mathfrak{x}$ is an automorphism of V if and only if $\mathfrak{a}$ is an idèle.

At present we shall consider idèles only in this role. Later we shall study the multiplicative group of idèles as a group in its own right, with its own topology, as the restricted direct product of the groups $k_{\mathfrak{p}}^{\times}$ relative to the subgroups $u_{\mathfrak{p}}$.

To answer the original question concerning the transformation of the measure under these automorphisms we state

Lemma 4.1.2: For an idèle, $\mathfrak{m}$,

$$d(\mathfrak{m}\mathfrak{x}) = |\mathfrak{m}|\, d\mathfrak{x}, \text{ where}$$

$$|\mathfrak{m}| = \prod_{\mathfrak{p}} |\mathfrak{m}_{\mathfrak{p}}|_{\mathfrak{p}} \qquad \text{(really a finite product).}$$

Proof: If $N = \prod_{\mathfrak{p}} N_{\mathfrak{p}}$ is a compact neighborhood of 0 in V, then by theorem 3.3.1 and lemma 2.2.5

$$\int_N d\mathfrak{x} = \prod_{\mathfrak{p}} \int_{N_{\mathfrak{p}}} d\mathfrak{x}_{\mathfrak{p}}, \text{ and } \int_{\mathfrak{m}N} d\mathfrak{x} = \prod_{\mathfrak{p}} \int_{\mathfrak{m}_{\mathfrak{p}}N_{\mathfrak{p}}} d\mathfrak{x}_{\mathfrak{p}} = \prod_{\mathfrak{p}} |\mathfrak{m}_{\mathfrak{p}}|_{\mathfrak{p}} \int_{N_{\mathfrak{p}}} d\mathfrak{x}_{\mathfrak{p}}$$

The last, and most important thing we must do in our preliminary discussion of V is to see how the field k is imbedded in V. We identify the element $\xi \in k$ with the valuation vector $\xi = (\xi, \xi, \ldots, \xi, \ldots)$ having all components equal to ξ, and view k as subgroup of V. What kind of subgroup is it?

Lemma 4.1.3: If S_∞ denotes the set of archimedian primes of k, then 1.) $k \cap V_{S_\infty} = \mathfrak{o}$, the ring of algebraic integers in k, and 2.) $k + V_{S_\infty} = V$.

Proof: 1.) This is simply the statement that an element $\xi \in k$ is an algebraic integer if and only if it is an integer at all finite primes.

2.) $k + V_{S_\infty} = V$ means: given any $\mathfrak{x} \in V$, there exists a $\xi \in k$ approximating it in the sense that $\xi - \mathfrak{x}_{\mathfrak{p}} \in \mathfrak{o}_{\mathfrak{p}}$ for all finite $\mathfrak{p}$. Such a ξ can be found by solving simultaneous consequences in $\mathfrak{o}$. The existence of a solution is guaranteed by the Chinese Remainder theorem.

Let now $\overset{\infty}{V}$ denote the "infinite part" of V, i.e. the cartesian product $\prod_{\mathfrak{p}\in S_\infty} k_{\mathfrak{p}}$ of the archimedian completions of k. If a generating equation for k over the rational field has r_1 real roots and r_2 pairs of conjugate complex roots, then $\overset{\infty}{V}$ is the product of r_1 real lines and r_2 complex planes. As such it is naturally a vector space over the real numbers of dimension $n = r_1 + 2r_2$ = absolute degree of k. For any $\mathfrak{X} \in V$ denote by $\overset{\infty}{\mathfrak{X}}$ the projection $\overset{\infty}{\mathfrak{X}} = (\dots, \mathfrak{X}_{\mathfrak{p}}, \dots)_{\mathfrak{p}\in S_\infty}$ of $\mathfrak{X}$ on $\overset{\infty}{V}$.

<u>Lemma 4.1.4</u>: If $\{\omega_1, \omega_2, \dots, \omega_n\}$ is a minimal basis for the ring of integers $\mathfrak{o}$ of k over the rational integers, then $\{\overset{\infty}{\omega}_1, \overset{\infty}{\omega}_2, \dots, \overset{\infty}{\omega}_n\}$ is a basis for the vector space $\overset{\infty}{V}$ over the real numbers. The parallelotope $\overset{\infty}{D}$ spanned by this basis ($\overset{\infty}{D}$ = set of all $\overset{\infty}{\mathfrak{X}} = \sum_{\nu=1}^{n} x_\nu \overset{\infty}{\omega}_\nu$ with $0 \leq x_\nu < 1$) has the volume $\sqrt{|d|}$ (where $d = (\det(\omega_i^{(j)}))^2$ = absolute discriminant of k) if measured in the measure $d\overset{\infty}{\mathfrak{X}} = \prod_{\mathfrak{p}\in S_\infty} d\mathfrak{X}_{\mathfrak{p}}$ which is natural in our set-up.

<u>Proof</u>: The projection $\xi \to \overset{\infty}{\xi}$ of k into $\overset{\infty}{V}$ is just the classical imbedding of a number field into n-space. The reader will remember the classical argument which runs: k separable $\Rightarrow d = (\det(\omega_i^{(j)}))^2 \neq 0$ $\Rightarrow \{\overset{\infty}{\omega}_1, \overset{\infty}{\omega}_2, \dots, \overset{\infty}{\omega}_n\}$ linearly independent, and (with a simple determinant computation) $\overset{\infty}{D}$ has volume $\frac{1}{2^{r_2}}\sqrt{|d|}$. For us the volume is 2^{r_2} times as much because we have chosen for complex $\mathfrak{p}$ a measure which is twice the ordinary measure in the complex plane.

<u>Definition 4.1.2</u>: The additive fundamental domain $D \subset V$ is the set of all $\mathfrak{X}$ such that $\mathfrak{X} \in V_{S_\infty}$ and $\overset{\infty}{\mathfrak{X}} \in \overset{\infty}{D}$.

<u>Theorem 4.1.3</u>: 1.) D deserves its name because any vector $\mathfrak{X} \in V$ is congruent to one and only one vector of D modulo the field elements ξ. In other words, $V = \bigcup_{\xi\in k} (\xi + D)$, a disjoint union.

2.) D has measure 1.

Proof: 1.) Starting with an arbitrary $\mathfrak{x} \in V$ we can bring it into V_{S_∞} by the addition of a field element which is unique mod $\mathfrak{v}$ (lemma 4.1.3). Once in V_{S_∞} we can find a unique element of $\mathfrak{v}$, by the addition of which we can stay in V_{S_∞} and adjust the infinite components so that they lie in $\overset{\infty}{\mathfrak{D}}$. (lemma 4.1.4).

2.) To compute the measure of D, notice that $D \subset V_{S_\infty}$ and $D = \overset{\infty}{\mathfrak{D}} \times V^{S_\infty}$. Therefore

$$\int_D d\mathfrak{x} = \int_D d\mathfrak{x}_{S_\infty} = \int_{\overset{\infty}{\mathfrak{D}} \times V^{S_\infty}} d\overset{\infty}{\mathfrak{x}} \cdot d\mathfrak{x}^{S_\infty} = \int_{\overset{\infty}{\mathfrak{D}}} d\overset{\infty}{\mathfrak{x}} \cdot \int_{V^{S_\infty}} d\mathfrak{x}^{S_\infty} = \sqrt{|d|} \cdot \prod_{\mathfrak{p} \notin S_\infty} (N_\mathfrak{p} \mathfrak{d}_\mathfrak{p})^{-\frac{1}{2}}$$

Now since the discriminant d (as ideal) is the norm of the absolute different $\mathfrak{d}$ of k, and since $\mathfrak{d}$ is the product of the local differents $\mathfrak{d}_\mathfrak{p}$, we have $|d| = \prod_{\mathfrak{p} \notin S_\infty} N_\mathfrak{p} \mathfrak{d}_\mathfrak{p}$. Therefore the measure which we have computed is 1.

Corollary 4.1.1: k is a discrete subgroup of V. The factor group V mod k is compact.

Proof: k is discrete, since D has an interior. V mod k is compact, since D is relatively compact.

Lemma 4.1.5: $\Lambda(\xi) = 0$ for all $\xi \in k$.

Proof:

$$\Lambda(\xi) = \sum_\mathfrak{p} \Lambda_\mathfrak{p}(\xi) = \sum_\mathfrak{p} \lambda_p(S_\mathfrak{p}(\xi)) = \sum_p \lambda_p\Big(\sum_{\mathfrak{p}|p} S_\mathfrak{p}(\xi)\Big) = \sum_p \lambda_p S(\xi)$$

because "the trace is the sum of the local traces". Since $S(\xi)$ is a rational number, the problem is reduced to proving that $\sum_p \lambda_p(x) \equiv 0$ (mod 1) for rational x. This we do by observing that the rational number $\sum_p \lambda_p(x)$ is integral with respect to each fixed rational prime q. Namely

$$\sum_p \lambda_p(x) = \Big(\sum_{p \neq q, p_\infty} \lambda_p(x)\Big) + \lambda_q(x) + \lambda_{p_\infty}(x) = \Big(\sum_{p \neq q, p_\infty} \lambda_p(x)\Big) + (\lambda_q(x) - x)$$

expresses $\lambda(x)$ as sum of q-adic integers.

Theorem 4.1.4: $k^* = k$; that is $\Lambda(\mathfrak{y}\xi) = 0$ for all $\xi \iff \mathfrak{y} \in k$.

Proof: Since k^* is the character group of the compact factor group V mod k, k^* is discrete. k^* contains k according to the preceding lemma, and therefore we may consider the factor group k^* mod k. As discrete subgroup of the compact group V mod k, k^* mod k is a finite group. But since it is a priori clear that k^* is a vector space over k, and since k is not a finite field, the index $(k^*:k)$ cannot be finite unless it is 1.

4.2 Riemann-Roch Theorem: We shall call a function $\varphi(\mathfrak{y})$ periodic if $\varphi(\mathfrak{y}+\xi) = \varphi(\mathfrak{y})$ for all $\xi \in k$. The periodic functions represent in a natural way all functions on the compact factor group V mod k. $\varphi(\mathfrak{y})$ represents a continuous function on V mod k if and only if it is itself continuous on V.

Lemma 4.2.1: If $\varphi(\mathfrak{y})$ is continuous and periodic, then $\int_D \varphi(\mathfrak{y})d\mathfrak{y}$ is equal to:

> the integral over the factor group V mod k of the function on that group which $\varphi(\mathfrak{y})$ represents with respect to that Haar measure on V mod k which gives the whole group V mod k the measure 1.

Proof: Define $I(\varphi) = \int_D \varphi(\mathfrak{y})d\mathfrak{y}$ and consider it as functional on $L(V \bmod k)$. Observe that it has the properties characterizing the Haar integral. (To check invariance under translation merely requires breaking D up into a disjoint sum of its intersections with a translation of itself). The functional is normed to 1 because $\int_D d\mathfrak{y} = 1$.

k is naturally the character group of V mod k in view of theorem 4.1.4. The Fourier transform, $\hat{\varphi}(\xi)$, of the continuous function on V mod k which is represented by $\varphi(\mathfrak{y})$ is

$$\hat{\varphi}(\xi) = \int_D \varphi(\psi) e^{-2\pi i \Lambda(\xi\psi)} d\psi.$$

<u>Lemma 4.2.2</u>: If $\varphi(\psi)$ is continuous and periodic and $\sum_{\xi\in k} |\hat{\varphi}(\xi)| < \infty$, then

$$\varphi(\psi) = \sum_{\xi\in k} \hat{\varphi}(\xi) e^{2\pi i \Lambda(\psi\xi)}$$

<u>Proof</u>: The hypothesis $\sum_{\xi\in k} |\varphi(\xi)| < \infty$ means that the Fourier transform $\hat{\varphi}(\xi)$ is summable on k, guaranteeing that the inversion formula holds. The asserted equality is simply the inversion formula explicitly written out.

<u>Lemma 4.2.3</u>: If $f(\psi)$ is continuous, $\in L_1(V)$, and $\sum_{\eta\in k} f(\psi+\eta)$ is uniformly convergent in ψ (convergence means absolute convergence because k is not ordered in any way), then for the resulting continuous periodic function $\varphi(\psi) = \sum_{\eta\in k} f(\psi+\eta)$ we have $\hat{\varphi}(\xi) = \hat{f}(\xi)$.

<u>Proof</u>:

$$\begin{aligned}
\hat{\varphi}(\xi) &= \int_D \varphi(\psi) e^{-2\pi i \Lambda(\psi\xi)} d\psi \\
&= \int_D \Big(\sum_{\eta\in k} f(\psi+\eta)\Big) e^{-2\pi i \Lambda(\psi\xi)} d\psi \\
&= \sum_{\eta\in k} \int_D f(\psi+\eta) e^{-2\pi i \Lambda(\psi\xi)} d\psi
\end{aligned}$$

(The interchange is justified because we assumed the convergence to be uniform on D, and D has finite measure).

$$\begin{aligned}
&= \sum_{\eta\in k} \int_{\eta+D} f(\psi) e^{-2\pi i \Lambda(\psi\xi - \eta\xi)} d\psi \\
&= \sum_{\eta\in k} \int_{\eta+D} f(\psi) e^{-2\pi i \Lambda(\psi\xi)} d\psi \qquad (\Lambda(\eta\xi) = 0) \\
&= \int f(\psi) e^{-2\pi i \Lambda(\psi\xi)} d\psi \\
&= \hat{f}(\xi).
\end{aligned}$$

Combining the last two lemmas 4.2.2 and 4.2.3, and putting $x = 0$ in the assertion of lemma 4.2.2 we obtain

Lemma 4.2.4: (Poisson Formula) If $f(\mathfrak{y})$ satisfies the conditions:

1.) $f(\mathfrak{y})$ continuous, $\in L_1(V)$;

2.) $\sum_{\xi \in k} f(\mathfrak{y}+\xi)$ uniformly convergent in $\mathfrak{y}$;

3.) $\sum_{\xi \in k} |\hat{f}(\xi)|$ convergent;

Then

$$\sum_{\xi \in k} \hat{f}(\xi) = \sum_{\xi \in k} f(\xi).$$

If we replace $f(\mathfrak{y})$ by $f(\mathfrak{m}\mathfrak{y})$ ($\mathfrak{m}$ an idèle) we obtain a theorem which may be looked upon as the number theoretic analogue of the Riemann-Roch theorem.

Theorem 4.2.1: (Riemann-Roch Theorem) If $f(\mathfrak{y})$ satisfies the conditions:

1.) $f(\mathfrak{y})$ continuous, $\in L_1(V)$;

2.) $\sum_{\xi \in k} f(\mathfrak{m}(\mathfrak{y}+\xi))$ convergent for all idèles $\mathfrak{m}$ and valuation vectors $\mathfrak{y}$, uniformly in $\mathfrak{y}$;

3.) $\sum_{\xi \in k} |\hat{f}(\mathfrak{m}\xi)|$ convergent for all idèles $\mathfrak{m}$;

then

$$\frac{1}{|\mathfrak{m}|} \sum_{\xi \in k} \hat{f}\left(\frac{\xi}{\mathfrak{m}}\right) = \sum_{\xi \in k} f(\mathfrak{m}\xi).$$

Proof: The function $g(\mathfrak{y}) = f(\mathfrak{m}\mathfrak{y})$ satisfies the conditions of the preceding lemma because

$$\hat{g}(\mathfrak{y}) = \int f(\mathfrak{m}\mathfrak{x})\, e^{-2\pi i \Lambda(\mathfrak{y}\mathfrak{x})}\, d\mathfrak{x}$$

$$= \frac{1}{|\mathfrak{m}|} \int f(\mathfrak{x})\, e^{-2\pi i \Lambda\left(\frac{\mathfrak{y}\mathfrak{x}}{\mathfrak{m}}\right)}\, d\mathfrak{x}$$

(Under the transformation $\mathfrak{x} \to \mathfrak{x}/\mathfrak{m}$, $d\mathfrak{x} \to d\mathfrak{x}/|\mathfrak{m}|$).

$$= \frac{1}{|\mathfrak{m}|} \hat{f}\left(\frac{\mathfrak{y}}{\mathfrak{m}}\right).$$

We may therefore conclude

$$\sum_{\xi \in k} \hat{g}(\xi) = \sum_{\xi \in k} g(\xi), \text{ that is, } \frac{1}{|\mathfrak{m}|} \sum_{\xi \in k} \hat{f}\left(\frac{\xi}{\mathfrak{m}}\right) = \sum_{\xi \in k} f(\mathfrak{m}\xi)$$

It is amusing to remark that, had we never bothered to compute the exact measure of D, we would now know it is 1. For we could have carried out all the arguments of this section with an unknown measure, say $\mu(D)$, of D. The only change would be that in order to have the inversion formula of lemma 4.2.2 we would have to have given each element of k the weight $1/\mu(D)$. The Poisson Formula would then have read,

$$\frac{1}{\mu(D)} \sum_{\xi \in k} \hat{f}(\xi) = \sum_{\xi \in k} f(\xi)$$

Iteration of this would yield $(\mu(D))^2 = 1$, therefore $\mu(D) = 1$!

4.3 Multiplicative Theory. In this section we shall discuss the basic features of the multiplicative group of idèles.

Definition 4.3.1: The multiplicative group, I, of idèles is the restricted direct product of the groups $k_{\mathfrak{p}}^{\times}$ relative to the subgroups $\mathfrak{u}_{\mathfrak{p}}$.

We shall denote the generic idèle by $\mathfrak{a} = (\ldots, \mathfrak{a}_{\mathfrak{p}}, \ldots)$. The name idèle is explained (at least partly explained!) by the fact that the idèle group may be considered as a refinement of the ideal group of k. For if we associate with an idèle $\mathfrak{a}$ the ideal $\varphi(\mathfrak{a}) = \prod_{\mathfrak{p} \notin S_\infty} \mathfrak{p}^{\operatorname{ord}_{\mathfrak{p}} \mathfrak{a}_{\mathfrak{p}}}$, then the map $\mathfrak{a} \to \varphi(\mathfrak{a})$ is obviously a continuous homomorphism of the idèle group onto the discrete group of ideals of k. Since the kernel of this homomorphism is I_{S_∞}, we may say that an idèle is a refinement of an ideal in two ways. First, the archimedean primes figure in its make-up, and second, it takes into account the units at the discrete primes.

Concerning quasi-characters of I, we can only state, according to § 3.2, that the general quasi-character $c(\mathfrak{a})$ is of the form $c(\mathfrak{a}) = \prod_{\mathfrak{p}} c_{\mathfrak{p}}(\mathfrak{a}_{\mathfrak{p}})$, where $c_{\mathfrak{p}}(\mathfrak{a}_{\mathfrak{p}})$ is a local quasi-character (described in § 2.3) and $c_{\mathfrak{p}}(\mathfrak{a}_{\mathfrak{p}})$ is unramified at almost all $\mathfrak{p}$.

For a measure, $d\mathfrak{a}$, on I we shall of course choose $d\mathfrak{a} = \prod_{\mathfrak{p}} d\mathfrak{a}_{\mathfrak{p}}$ the $d\mathfrak{a}_{\mathfrak{p}}$ being the local multiplicative measure defined in §2.3.

We can do nothing really significant with the idèle group until we imbed the multiplicative group $k^{\times}$ of k in it, by identifying the element $\alpha \in k^{\times}$ with the idèle $\alpha = (\alpha, \alpha, \ldots, \alpha, \ldots)$. Throughout the remainder of this section our discussion will center about the structure of I relative to the subgroup $k^{\times}$. The first fact to notice is that the ideal $\varphi(\alpha)$ associated with an idèle $\alpha \in k^{\times}$ is the principal ideal $\alpha\mathfrak{o}$ generated by α, as it should be. Next we have the "product formula" for elements $\alpha \in k^{\times}$. Though this is well-known, we state it formally in a theorem in order to present an amusing proof.

<u>Theorem 4.3.1:</u> $|\mathfrak{a}| \; (= \prod_{\mathfrak{p}} |\mathfrak{a}|_{\mathfrak{p}}) = 1$ for $\alpha \in k^{\times}$.

<u>Proof:</u> According to lemma 4.1.2 the (additive) measure of αD is $|\alpha|$ times the measure of D. Since $\alpha k^{+} = k^{+}$, αD would serve as additive fundamental domain just as well as D. From this it is intuitively clear that αD has the same measure as D and therefore $|\alpha| = 1$. To make a formal proof one has simply to chop up D and αD into congruent pieces of the form $D \cap (\xi + \alpha D)$ and $(-\xi + D) \cap \alpha D$ respectively, ξ running through k.

This theorem reminds us to mention explicitly the continuous homomorphism $\mathfrak{a} \to |\mathfrak{a}| = \prod_{\mathfrak{p}} |\mathfrak{a}|_{\mathfrak{p}}$ of I onto the multiplicative group of positive real numbers. The kernel is a closed subgroup of I which will play an important role. We denote this subgroup by J, and its generic element (idèle of absolute value 1) by $\hat{\mathfrak{b}}$.

It will be convenient (although it is aesthetically disturbing and not really necessary) to select arbitrarily a subgroup T of I with which we can write $I = T \times J$ (direct product). To this effect we choose at random one of the archimedean primes of k - call it $\mathfrak{p}_0$ - and let T be the subgroup of all idèles such that $\mathfrak{a}_{\mathfrak{p}_0} > 0$ and $\mathfrak{a}_{\mathfrak{p}} = 1$ for $\mathfrak{p} \neq \mathfrak{p}_0$. Such an idèle is obviously uniquely determined by its absolute value; indeed the map $\mathfrak{a} \to |\mathfrak{a}|$, restricted to T, is an isomorphism between T and the multiplicative group of positive real numbers, and it will cause no confusion if we denote an idèle of T simply by the real number which is its absolute value. Thus a real number $t > 0$ also stands either for the idèle $(t, 1, 1, \dots)$ or for the idèle $(\sqrt{t}, 1, 1, \dots)$, according to whether $\mathfrak{p}_0$ is real or complex, if we write the $\mathfrak{p}_0$- component first. Since we can write any idèle $\mathfrak{a}$ uniquely in the form $\mathfrak{a} = |\mathfrak{a}|\mathfrak{b}$ with $|\mathfrak{a}| \in T$ and $\mathfrak{b} = \mathfrak{a}|\mathfrak{a}|^{-1} \in J$, it is clear that $I = T \times J$ (direct product).

In order to select a fixed measure $d\mathfrak{b}$ on J we take on T the measure $d t = dt/t$ and require $d\mathfrak{a} = dt \cdot d\mathfrak{b}$. Then for computational purposes we have (in the sense of Fubini) the formulas

$$\int f(\mathfrak{a})d\mathfrak{a} = \int_0^\infty \Big[\int_J f(t\mathfrak{b})\, d\mathfrak{b} \Big] \frac{dt}{t} = \int_J \Big[\int_0^\infty f(t\mathfrak{b}) \frac{dt}{t} \Big] d\mathfrak{b}$$

for a summable idèle function $f(\mathfrak{a})$.

The product formula means that $k^\times \subset J$, and we wish now to describe a "fundamental domain" for J mod $k^\times$. The mapping of ideles onto ideals allows us to descend to the subgroup $J_{S_\infty} = J \cap I_{S_\infty}$. To study J_{S_∞} we map the idèles $\mathfrak{b} \in J_{S_\infty}$ onto vectors $\ell(\mathfrak{b}) = (\dots, \log|\mathfrak{b}|_{\mathfrak{p}}, \dots)_{\mathfrak{p} \in S'_\infty}$ having one component, $\log|\mathfrak{b}|_{\mathfrak{p}}$ for each archimedean prime except $\mathfrak{p}_0$. (This set of $r = r_1 + r_2 - 1$ primes is denoted by S'_∞.) It is obvious that the map $\mathfrak{b} \to \ell(\mathfrak{b})$ is a continuous homomorphism of J_{S_∞} onto the additive group of Euclidean r-space. The onto-ness results from the fact that although the infinite components of an idèle $\mathfrak{b} \in J_{S_\infty}$ are constrained by the condition $\prod_{\mathfrak{p}} |\mathfrak{b}|_{\mathfrak{p}} = \prod_{\mathfrak{p} \in S_\infty} |\mathfrak{b}|_{\mathfrak{p}} = 1$, they are completely free in the set S'_∞ since we can adjust the $\mathfrak{p}_0$ component

$k^\times \cap J_{S_\infty}$ is the group of all elements $\varepsilon \in k^\times$ which are units at all finite primes; that is, which are units of the ring $\mathfrak{o}$. The units ζ for which $\ell(\zeta) = 0$ are the roots of unity in k and form a finite cyclic group. It is proved classically that the group of units ε, modulo the group of roots of unity ζ, is a free abelian group on r generators. This proof is effected by showing that the images $\ell(\varepsilon)$ of units ε form a lattice of highest dimension in the r-space.

If, therefore, $\{\varepsilon_i\}_{1\le i\le r}$ is a basis for the group of units modulo roots of unity, the vectors $\ell(\varepsilon_i)$ are a basis for the r-space over the real numbers and we may write for any $\mathfrak{b} \in J_{S_\infty}$, $\ell(\mathfrak{b}) = \sum_{\nu=1}^{r} x_\nu \ell(\varepsilon_\nu)$, with unique real numbers x_ν. Call P the parallelotope in the R-space spanned by the vectors $\ell(\varepsilon_i)$; that is, the set of all vectors $\sum_{\nu=1}^{r} x_\nu \ell(\varepsilon_\nu)$ with $0 \le x_\nu < 1$. Call Q the "unit cube" in the r-space; that is the set of all vectors $(\dots, x_{\mathfrak{p}}, \dots)_{\mathfrak{p}\in S'_\infty}$ with $0 \le x_{\mathfrak{p}} < 1$.

Lemma 4.3.1:

$$\int_{\ell^{-1}(P)} d\mathfrak{b} = \frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{|d|}} R, \text{ where}$$

$\ell^{-1}(P)$ is the set of all $\mathfrak{b} \in J_{S_\infty}$ such that $\ell(\mathfrak{b}) \in P$, and $R = \pm\det(\log |\varepsilon_i|_{\mathfrak{p}})_{\substack{1\le i\le r\\ \mathfrak{p}\in S'_\infty}}$ is the regulator of k.

Proof: Because ℓ is a homomorphism,

$$\frac{\text{measure of } \ell^{-1}(P)}{\text{measure of } \ell^{-1}(Q)} = \frac{\text{volume of } P}{\text{volume of } Q} = \pm\det(\log |\varepsilon_i|_{\mathfrak{p}}) = R,$$

and we have only to show $\displaystyle\int_{\ell^{-1}(Q)} d\mathfrak{b} = \frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{|d|}}$.

$\ell^{-1}(Q)$ is the set of all $\mathfrak{b} \in J_{S_\infty}$ with $1 \le |\mathfrak{b}|_{\mathfrak{p}} < e$ for $\mathfrak{p} \in S'_\infty$.

Let Q^* be the set of all $\mathfrak{a} \in I_{S_\infty}$ with $1 \le |\mathfrak{a}|_{\mathfrak{p}} < e$ for $\mathfrak{p} \in S_\infty$.

Then $\displaystyle\int_{Q^*} d\mathfrak{a} = \int_J \Big[\int_{t\mathfrak{b}\in Q^*} \frac{dt}{t}\Big] d\mathfrak{b} = \int_{\ell^{-1}(Q)} \Big[\int_{|\mathfrak{b}|^{-1}_{\mathfrak{p}_0}}^{e|\mathfrak{b}|^{-1}_{\mathfrak{p}_0}} \frac{dt}{t}\Big] d\mathfrak{b} = \int_{\ell^{-1}(Q)} d\mathfrak{b}$,

because $t\mathfrak{b} \in Q^* \iff \mathfrak{b} \in \ell^{-1}(Q)$ and $1 \le |t\mathfrak{b}|_{\mathfrak{p}} < e$. We have therefore only to show that $\displaystyle\int_{Q^*} d\mathfrak{w} = \frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{|d|}}$.

Write Q^* as the cartesian product $Q^* = \prod_{\mathfrak{y}\in S_\infty} Q^*_{\mathfrak{y}} \times I^{S_\infty}$, where $Q^*_{\mathfrak{y}}$ is the set of all $\mathfrak{a}_{\mathfrak{y}} \in k^\times_{\mathfrak{y}}$ such that $1 \le |\mathfrak{a}_{\mathfrak{y}}|_{\mathfrak{y}} \le e$, for $\mathfrak{y} \in S_\infty$. Then

$$\int_{Q^*} d\mathfrak{w} = \prod_{\mathfrak{y}\in S_\infty} \int_{Q^*_{\mathfrak{y}}} d\mathfrak{w}_{\mathfrak{y}} \cdot \int_{I^{S_\infty}} d\mathfrak{w}^{S_\infty} = \frac{2^{r_1}(2\pi)^{r_2}}{\sqrt{|d|}},$$ because

for $\mathfrak{y}$ real, $$\int_{Q^*_{\mathfrak{y}}} d\mathfrak{a}_{\mathfrak{y}} = \left[\int_{-e}^{-1} + \int_1^e\right]\frac{dx}{|x|} = 2\int_1^e \frac{dx}{x} = 2,$$

for $\mathfrak{y}$ complex, $$\int_{Q^*_{\mathfrak{y}}} d\mathfrak{w}_{\mathfrak{y}} = \int_0^{2\pi}\int_1^{\sqrt{e}} \frac{2\,r\,dr\,d\theta}{r^2} = 2\pi,$$

and $$\int_{I^{S_\infty}} d\mathfrak{w}^{S_\infty} = \prod_{\mathfrak{y}\notin S_\infty} \int_{U_{\mathfrak{y}}} d\mathfrak{w}_{\mathfrak{y}} = \prod_{\mathfrak{y}\notin S_\infty} (N_{\mathfrak{y}}\delta_{\mathfrak{y}})^{-\frac{1}{2}} = \frac{1}{\sqrt{|d|}}.$$

<u>Definition 4.3.2</u>: Let h be the class number of k, and select ideals $\mathfrak{b}^{(1)}, \ldots, \mathfrak{b}^{(h)} \in J$ such that the corresponding ideals $\varphi(\mathfrak{b}^{(1)}), \ldots, \varphi(\mathfrak{b}^{(h)})$ represent the different ideal classes. Let w be the number of roots of unity in k. Let E_0 be the subset of all $\mathfrak{b} \in \lambda^{-1}(P)$ (see preceding lemma) such that $0 \le \arg \mathfrak{b}_{\mathfrak{y}_0} < \frac{2\pi}{w}$. We define the multiplicative fundamental domain, E, for J mod $k^\times$ to be

$$E = E_0\mathfrak{b}^{(1)} \cup E_0\mathfrak{b}^{(2)} \cup \ldots \cup E_0\mathfrak{b}^{(h)}.$$

<u>Theorem 4.3.2</u>: 1.) $J = \bigcup_\alpha \alpha E$, a disjoint union.

2.) $$\int_E d\mathfrak{b} = \frac{2^{r_1}(2\pi)^{r_2} h R}{\sqrt{|d|}\, w}.$$

<u>Proof</u>: 1.) Starting with any idèle $\mathfrak{b} \in J$ we can change it into an idèle which represents a principal ideal by dividing it by a uniquely determined $\mathfrak{b}^{(i)}$. If this principal ideal is $\alpha\mathfrak{o}$ (α uniquely determined modulo units), multiplication by α^{-1} brings us to an idèle of J representing the ideal $\mathfrak{o}$ — therefore into J_{S_∞}. Once in J_{S_∞} we can find a unique power product of the fundamental units ε_i which lands us in $\lambda^{-1}(P)$, with only a root of unity, ζ, at our disposal. This ζ is exactly what we need to adjust the argument of the $\mathfrak{y}_0$ component to be in the interval $[0, \frac{2\pi}{w})$. Lo and behold we are in E_0! For our original idèle $\mathfrak{b}$ we have found a unique $\alpha \in k^\times$ and a unique $\mathfrak{b}^{(i)}$ such that $\mathfrak{b} \in \alpha\mathfrak{b}^{(i)}E_0$.

2.) (measure E) = $h\cdot$(measure E_0) = $\frac{h}{w}$(measure $\ell^{-1}(P)$) = $\dfrac{2^{r_1}(2\pi)^{r_2} h R}{\sqrt{|d|}\; w}$

according to the two disjoint decompositions

$$E = \bigcup_{\nu=1}^{h} \mathfrak{b}^{(\nu)} E_0 \,, \qquad \ell^{-1}(P) = \bigcup_{\zeta} \zeta E_0$$

and the preceding lemma.

<u>Corollary 4.3.1</u>: $k^\times$ is a discrete subgroup of J (therefore of I). J mod $k^\times$ is compact.

<u>Proof</u>: One sees easily that E has an interior in J. On the other hand, E is contained in a compact.

We shall really be interested not in all quasi-characters of I, but only in those which are trivial on $k^\times$. From now on when we use the word quasi-character we mean one of this type. Let us close our introduction to the idèle group with a few remarks about these quasi-characters.

The first thing to notice is that on the subgroup J, a quasi-character is a character; i.e. $|c(\mathfrak{b})| = 1$ for all $\mathfrak{b} \in J$, because J mod $k^\times$ is compact.

Next we mention that the quasi-characters which are trivial on J are exactly those of the form $c(\mathfrak{a}) = |\mathfrak{a}|^s$, where s is a complex number uniquely determined by $c(\mathfrak{a})$. For if $c(\mathfrak{a})$ is trivial on J, then $c(\mathfrak{a})$ depends only on $|\mathfrak{a}|$, and in this dependence is a continuous multiplicative map of the positive real numbers into the complex numbers. Such a map is of the form $t \to t^s$ as is well-known.

To each quasi-character $c(\mathfrak{a})$ there exists a unique real number σ such that $|c(\mathfrak{a})| = |\mathfrak{a}|^\sigma$. Namely, $|c(\mathfrak{a})|$ is a quasi-character which is trivial on J. Therefore $|c(\mathfrak{a})| = |\mathfrak{a}|^s$, for some complex s. Since $|c(\mathfrak{a})| > 0$, s is real. We call σ the exponent of c. A quasi-character is a character if and only if its exponent is 0.

4.4 <u>The ζ-Functions; Functional Equation.</u> In this section $f(\mathfrak{x})$ will denote a complex-valued function of valuation vectors; $f(\mathfrak{a})$ its restriction to idèles. We let $\mathfrak{Z}$ denote the class of all functions $f(\mathfrak{x})$ satisfying the three conditions:

$\mathfrak{Z}_1$) $f(\mathfrak{x})$, and $\hat{f}(\mathfrak{x})$ are continuous, $\in L_1(V)$; i.e. $f(\mathfrak{x}) \in \mathfrak{W}_1(V)$.

$\mathfrak{Z}_2$) $\sum_{\xi \in k} f(\mathfrak{a}(\mathfrak{x}+\xi))$ and $\sum_{\xi \in k} \hat{f}(\mathfrak{a}(\mathfrak{x}+\xi))$ are both convergent for each idèle $\mathfrak{a}$ and vector $\mathfrak{x}$, the convergence being uniform in the pair $(\mathfrak{a}, \mathfrak{x})$ for $\mathfrak{x}$ ranging over V and $\mathfrak{a}$ ranging over any fixed compact subset of I.

$\mathfrak{Z}_3$) $f(\mathfrak{a})|\mathfrak{a}|^{\sigma}$ and $\hat{f}(\mathfrak{a})|\mathfrak{a}|^{\sigma} \in L_1(I)$ for $\sigma > 1$.

(notice that if $f(\mathfrak{x})$ is continuous on V, then, a fortiori, $f(\mathfrak{a})$ is continuous on I, since the topology we have adopted in I is stronger than that which I would get as subspace of V).

In view of $\mathfrak{Z}_1$) and $\mathfrak{Z}_2$), the Riemann-Roch theorem is valid for functions of $\mathfrak{Z}$. The purpose of $\mathfrak{Z}_3$) is to enable us to define ζ-functions with them:

<u>Definition 4.4.1:</u> We associate with each $f \in \mathfrak{Z}$ a function $\zeta(f, c)$ of quasi-characters, defined for all quasi-characters c of exponent greater than 1 by

$$\zeta(f, c) = \int f(\mathfrak{a})\, c(\mathfrak{a})\, d\mathfrak{a}.$$

We call such a function a ζ-function of k.

Remember that we are now considering only those quasi-characters which are trivial on $k^{\times}$. These were discussed at the end of the preceding section, where the notion "exponent" is explained. If we call two quasi-characters which coincide on J equivalent, then an equivalence class of quasi-characters consists of all quasi-characters of the form $c(\mathfrak{a}) = c_0(\mathfrak{a})|\mathfrak{a}|^{s}$, where $c_0(\mathfrak{a})$ is a fixed representative of the class and s is a complex number uniquely determined by c.

Such a parametrization by the complex variable s allows us to view an equivalence class of quasi-characters as a Riemann surface, just as we did in the local theory (cf. § 2.4). It is obvious from their definition as an integral that the ζ-functions are regular in the domain of all quasi-characters of exponent greater than 1. (see the corresponding local lemma). What about analytic continuation???

<u>Main Theorem 4.4.1</u>: (Analytic Continuation and Functional Equation of the ζ-Functions). By analytic continuation we may extend the definition of any ζ-function $\zeta(f, c)$ to the domain of all quasi-characters. The extended function is single valued and regular, except at $c(\mathfrak{a}) = 1$ and $c(\mathfrak{a}) = |\mathfrak{a}|$ where it has simple poles with residues $-\kappa f(0)$ and $+\kappa \hat{f}(0)$, respectively ($\kappa = 2^{r_1}(2\pi)^{r_2} h R/(w\sqrt{|d|})$ = volume of the multiplicative fundamental domain). $\zeta(f, c)$ satisfies the functional equation

$$\zeta(f, c) = (\hat{f}, \hat{c}) ,$$

where $\hat{c}(\mathfrak{a}) = |\mathfrak{a}|\, c^{-1}(\mathfrak{a})$ as in the local theory.

<u>Proof</u>: For c of exponent greater than 1 we have $\zeta(f, c) = \int f(\mathfrak{a})\, c(\mathfrak{a})\, d\mathfrak{a} = \int_0^\infty \left[\int_J f(t\mathfrak{b}) c(t\mathfrak{b})\, d\mathfrak{b} \right] \frac{dt}{t} = \int_0^\infty \zeta_t(f, c) \frac{dt}{t}$, say. Here $\zeta_t(f, c) = \int_J f(t\mathfrak{b}) c(t\mathfrak{b})\, d\mathfrak{b}$ is absolutely convergent for c of any exponent, at least for almost all t, because it is convergent for some c, and $|c(t\mathfrak{b})| = t^{\text{exponent } c}$ is constant for $\mathfrak{b} \in J$. The essential step in our proof consists in using the Riemann-Roch theorem to establish a functional equation for $\zeta_t(f, c)$:

<u>Lemma A</u>: For all quasi-characters we have

$$\zeta_t(t, c) + f(0) \int_E c(t\mathfrak{b})\, d\mathfrak{b} = \zeta_{\frac{1}{t}}(\hat{f}, \hat{c}) + \hat{f}(0) \int_E \hat{c}(\tfrac{1}{t}\mathfrak{b})\, d\mathfrak{b}.$$

$$= \sum_{\alpha \in k^\times} \int_{\alpha E} f(t\mathfrak{z})\, c(t\mathfrak{z})\, d^\times\mathfrak{z} \; + \; f(0) \int_E c(t\mathfrak{z})\, d^\times\mathfrak{z}$$

(Because $J = \bigcup_\alpha \alpha E$, a disjoint union.)

$$= \sum_{\alpha \in k^\times} \int_E f(\alpha t\mathfrak{z})\, c(t\mathfrak{z})\, d^\times\mathfrak{z} \; + \; f(0) \int_E c(t\mathfrak{z})\, d^\times\mathfrak{z}$$

$(d^\times \alpha\mathfrak{z} = d^\times\mathfrak{z} \;\; ; \;\; c(\alpha t\mathfrak{z}) = c(t\mathfrak{z}) \quad)$

$$= \int_E \Big[\sum_{\alpha \in k^\times} f(\alpha t\mathfrak{z}) \Big] c(t\mathfrak{z})\, d^\times\mathfrak{z} \; + \; \int_E f(0)\, c(t\mathfrak{z})\, d^\times\mathfrak{z}$$

(By hypothesis $\mathfrak{Z}_2$.) for f, the sum is uniformly convergent for $\mathfrak{z}$ in the relatively compact subset E).

$$= \int_E \Big[\sum_{\xi \in k} f(\xi t\mathfrak{z}) \Big] c(t\mathfrak{z})\, d^\times\mathfrak{z}$$

$$= \int_E \Big[\sum_{\xi \in k} \hat{f}\Big(\frac{\xi}{t\mathfrak{z}}\Big) \Big] \frac{1}{|t\mathfrak{z}|}\, c(t\mathfrak{z})\, d^\times\mathfrak{z}$$

(Riemann-Roch theorem 4.2.1)

$$= \int_E \Big[\sum_{\xi \in k} \hat{f}\Big(\xi \tfrac{1}{t}\mathfrak{z}\Big) \Big] \hat{c}\Big(\tfrac{1}{t}\mathfrak{z}\Big)\, d^\times\mathfrak{z}$$

$(\mathfrak{z} \to \frac{1}{\mathfrak{z}} \; ; \; d^\times\mathfrak{z} \to d^\times\mathfrak{z} \;)$

Reversing the steps completes the proof.

Lemma E:

$$\int_E c(t\mathfrak{z})\, d^\times\mathfrak{z} = \begin{cases} \kappa\, t^s , & \text{if } c(\mathfrak{a}) = |\mathfrak{a}|^s; \\ 0 , & \text{if } c(\mathfrak{a}) \text{ is non-trivial on } J. \end{cases}$$

Proof:

$$\int_E c(t\mathfrak{z})\, d^\times\mathfrak{z} = c(t) \int_E c(\mathfrak{z})\, d^\times\mathfrak{z} \, .$$

$\int_E c(\mathfrak{z})\, d^\times\mathfrak{z}$ is the integral over the factor group J mod $k^\times$ of the character of this group which $c(\mathfrak{z})$ represents. Therefore it is either κ (= measure of E), or 0, according to whether $c(\mathfrak{a})$ is trivial on J or not. In the former case we must notice that $c(t) = |t|^s = t^s$.

To prove the theorem write, for c of exponent greater than 1,

$$\zeta(f,c) = \int_0^\infty \zeta_t(f,c)\frac{dt}{t} = \int_0^1 \zeta_t(f,c)\frac{dt}{t} + \int_1^\infty \zeta_t(f,c)\frac{dt}{t}.$$

The $\int_1^\infty$ is no problem. For it is equal to the integral of $f(\mathfrak{a})\, c(\mathfrak{a})\, d\mathfrak{a}$ over that half of I where $|\mathfrak{a}| \geq 1$. Therefore it converges the better, the less the exponent of c is; and since it converges for c of exponent greater than 1, it must converge for all c. Now, the point is that we can use lemma A (and the auxiliary lemma B) to transform the $\int_0^1$ into an $\int_1^\infty$, thereby obtaining an analytic expression for $\zeta(f, c)$ which will be good for all c. Namely:

$$\int_0^1 \zeta_t(f,c)\frac{dt}{t} = \int_0^1 \zeta_{\frac{1}{t}}(\hat{f},\hat{c})\frac{dt}{t} + \left\{\!\!\left\{ \int_0^1 \kappa \hat{f}(0)\left(\frac{1}{t}\right)^{1-s}\frac{dt}{t} - \int_0^1 \kappa f(0)\, t^s \frac{dt}{t} \right\}\!\!\right\},$$

where the expression $\{\{\cdots\cdots\}\}$ is to be included only if c is trivial on J, in which case we assume $c(\mathfrak{a}) = |\mathfrak{a}|^s$. We are still looking only at c of exponent greater than 1. If $c(\mathfrak{a}) = |\mathfrak{a}|^s$ this means $\mathrm{Re}(s) > 1$, which is just what is needed for the auxiliary integrals under the double bracket to make sense. Evaluating them and making the substitution $t \to \frac{1}{t}$ in the main part of the expression we obtain

$$\int_0^1 \zeta_t(f,c)\frac{dt}{t} = \int_1^\infty \zeta_t(\hat{f},\hat{c})\frac{dt}{t} + \left\{\!\!\left\{ \frac{\kappa \hat{f}(0)}{s-1} - \frac{\kappa f(0)}{s} \right\}\!\!\right\},$$

and therefore

$$\zeta(f,c) = \int_1^\infty \zeta_t(f,c)\frac{dt}{t} + \int_1^\infty \zeta_t(\hat{f},\hat{c})\frac{dt}{t} + \left\{\!\!\left\{ \frac{\kappa \hat{f}(0)}{s-1} - \frac{\kappa f(0)}{s} \right\}\!\!\right\}.$$

The two integrals are analytic for all c. This expression gives therefore the analytic continuation of $\zeta(f, c)$ to the domain of all quasi-characters. From it we can read off the poles and residues directly. Noticing that for $c(\mathfrak{a}) = |\mathfrak{a}|^s$, $\hat{c}(\mathfrak{a}) = |\mathfrak{a}|^{1-s}$, we see that even the form of the expression is unchanged by the substitution $(f, c) \to (\hat{f}, \hat{c})$. Therefore the functional equation

$$\zeta(f, c) = \zeta(\hat{f}, \hat{c})$$

holds. The Main Theorem is proved!

4.5 Comparison with the Classical Theory. We will now show that our theory is not without content, inasmuch as there do exist non-trivial ζ-functions. In fact we shall exhibit for each equivalence class C of quasi-characters an explicit function $f \in \mathfrak{Z}$ such that the corresponding ζ-function $\zeta(f, c)$ is non-trivial on C. These special ζ-functions will turn out to be, essentially, the classical ζ-functions and L-series. The analytic continuation and the functional equation for our ζ-functions will yield the same for the classical functions.

We can pattern our discussion after the computation of the special local ζ-functions in §2.5. There we treated the cases k real, k complex, and k $\mathfrak{p}$-adic. Now we treat the case

k in the large!

The Equivalence Classes of Quasi-Characters: According to a remark at the end of §4.3, each class of quasi-characters can be represented by a character. To describe the characters in detail, we will take an arbitrary, but fixed, finite set of primes, S, (containing at least all archimedean primes) and discuss the characters which are unramified outside S. A character of this type is nothing more nor less than a product

$$c(\mathfrak{a}) = \prod_{\mathfrak{p}} c_{\mathfrak{p}}(\mathfrak{a}_{\mathfrak{p}})$$

of local characters, $c_{\mathfrak{p}}$, satisfying the two conditions

1.) $c_{\mathfrak{p}}$ unramified outside S.

2.) $\prod_{\mathfrak{p}} c_{\mathfrak{p}}(\alpha) = 1$, for $\alpha \in k^{\times}$.

To construct such characters and express them in more concrete terms, we write for $\mathfrak{p} \in S$:

$$c_{\mathfrak{p}}(\mathfrak{a}_{\mathfrak{p}}) = \tilde{c}_{\mathfrak{p}}(\tilde{\mathfrak{a}}_{\mathfrak{p}}) \, |\mathfrak{a}_{\mathfrak{p}}|_{\mathfrak{p}}^{it_{\mathfrak{p}}},$$

$\tilde{c}_{\mathfrak{p}}$ being a character of $\mathfrak{u}_{\mathfrak{p}}$, $t_{\mathfrak{p}}$ a real number (cf. Theorem 2.3.1).

For $\mathfrak{p} \notin S$, we throw all the local characters together into a single character, say

$$c^*(\mathfrak{N}) = \prod_{\mathfrak{p} \notin S} c_{\mathfrak{p}}(\mathfrak{N}_{\mathfrak{p}}),$$

and interpret c^* as coming from an ideal character. Namely: The map $\mathfrak{N} \to \varphi_S(\mathfrak{N}) = \prod_{\mathfrak{p} \notin S} \mathfrak{p}^{\mathrm{ord}_{\mathfrak{p}} \mathfrak{N}}$ is a homomorphism of the idèle group onto the multiplicative group of ideals prime to S. Its kernel is I_S. $c^*(\mathfrak{N})$ is identity on I_S. We have therefore

$$c^*(\mathfrak{N}) = \chi(\varphi_S(\mathfrak{N})),$$

where χ is some character of the group of ideals prime to S. Our character $c(\mathfrak{N})$ is now written in the form

$$c(\mathfrak{N}) = \prod_{\mathfrak{p} \in S} \tilde{c}_{\mathfrak{p}}(\tilde{\mathfrak{N}}_{\mathfrak{p}}) \cdot \prod_{\mathfrak{p} \in S} |\mathfrak{N}|_{\mathfrak{p}}^{it_{\mathfrak{p}}} \cdot \chi(\varphi_S(\mathfrak{N})).$$

To construct such characters we must select our $\tilde{c}_{\mathfrak{p}}$, $t_{\mathfrak{p}}$ and χ such that $c(\alpha) = 1$, for $\alpha \in k^{\times}$. For this purpose we first look at the S-units, ε, of k, i.e. the elements of $k^{\times} \cap I_S$, for which $\varphi_S(\varepsilon) = \mathfrak{o}$. Assume S contains $m+1$ primes; let ε_0 be a primitive root of unity in k, and let $\{\varepsilon_1, \dots, \varepsilon_m\}$ be a basis for the free abelian group of S-units modulo roots of unity. For $c(\mathfrak{N})$ to be trivial on the S-units it is then necessary and sufficient that $c(\varepsilon_\nu) = 1$, $0 \leq \nu \leq m$. The requirement $c(\varepsilon_0) = 1$ is simply a condition on the $\tilde{c}_{\mathfrak{p}}$:

A) $\displaystyle\prod_{\mathfrak{p} \in S} \tilde{c}_{\mathfrak{p}}(\varepsilon_0) = 1.$

We therefore first select a set of $\tilde{c}_{\mathfrak{p}}$, for $\mathfrak{p} \in S$, which satisfies A. The requirements $c(\varepsilon_\nu) = 1$, $1 \leq \nu \leq m$, give conditions on the $t_{\mathfrak{p}}$:

$$\prod_{\mathfrak{p} \in S} |\varepsilon_\nu|_{\mathfrak{p}}^{it_{\mathfrak{p}}} = \prod_{\mathfrak{p} \in S} \tilde{c}_{\mathfrak{p}}^{-1}(\tilde{\varepsilon}_{\nu\mathfrak{p}}) \qquad 1 \leq \nu \leq m$$

which will be satisfied if and only if the numbers $t_{\mathfrak{p}}$ solve the real linear equations

B) $\displaystyle\sum_{\mathfrak{p} \in S} t_{\mathfrak{p}} \log |\varepsilon_\nu|_{\mathfrak{p}} = i \log \Big(\prod_{\mathfrak{p} \in S} \tilde{c}_{\mathfrak{p}}(\tilde{\varepsilon}_{\nu\mathfrak{p}}) \Big), \quad 1 \leq \nu \leq m$

for <u>some</u> value of the logarithms on the righthand side. We now select a set of values for those logarithms and a set of numbers $t_{\mathfrak{p}}$ solving the resulting equations B. Since, as is well-known, the rank of the matrix $(\log |\varepsilon_\nu|_{\mathfrak{p}})$ is m, there always exist solutions $t_{\mathfrak{p}}$. And since $\sum_{\mathfrak{p}\in S} \log |\varepsilon_\nu|_{\mathfrak{p}} = 0$ for all ν, the most general solution is then $t'_{\mathfrak{p}} = t_{\mathfrak{p}} + t$, for any t. While we are on the subject of existence and uniqueness of the $t_{\mathfrak{p}}$ we may remind the reader that if $\mathfrak{p}$ is archimedean, different $t_{\mathfrak{p}}$ give different local characters $c_{\mathfrak{p}} = \tilde{c}_{\mathfrak{p}} | \, |_{\mathfrak{p}}^{it_{\mathfrak{p}}}$; but if $\mathfrak{p}$ is discrete, those $t_{\mathfrak{p}}$ which are congruent mod $2\pi/\log N\mathfrak{p}$ give the same local $c_{\mathfrak{p}}$.

Having selected the $\tilde{c}_{\mathfrak{p}}$ and $t_{\mathfrak{p}}$, how much freedom is left for the ideal character χ? Not much. The requirement $c(\alpha) = 1$ for all $\alpha \in k^\times$ means that χ must satisfy the condition

$$\text{c)} \quad \chi(\varphi_S(\alpha)) = \prod_{\mathfrak{p}\in S} \tilde{c}_{\mathfrak{p}}^{-1}(\tilde{\alpha}_{\mathfrak{p}})\, |\alpha|_{\mathfrak{p}}^{-it_{\mathfrak{p}}}$$

for all ideals of the form $\varphi_S(\alpha)$, the ideals obtained from principal ideals by cancelling the powers of primes in S from their factorization. These ideals form a subgroup of finite index h_S (less than or equal to the class number h; $h_S = 1$ if S large enough) in the group of all ideals prime to S. Since the multiplicative function of α on the righthand side of condition C.) has been fixed up to be trivial on the S-units, it amounts to a character of this subgroup of ideals of the form $\varphi_S(\alpha)$. We must select χ to be one of the finite number h_S of extensions of this character to the group of all ideals prime to S.

<u>The Corresponding Functions of $\mathfrak{z}$</u>: Having selected a character

$$c(\mathfrak{n}) = \prod_{\mathfrak{p}} c_{\mathfrak{p}}(\mathfrak{n}_{\mathfrak{p}}) = \prod_{\mathfrak{p}\in S} \tilde{c}_{\mathfrak{p}}(\tilde{\mathfrak{n}}_{\mathfrak{p}})\, |\mathfrak{n}|_{\mathfrak{p}}^{it_{\mathfrak{p}}} \cdot \chi(\varphi_S(\mathfrak{n})),$$

unramified outside S, we wish to find a simple function $f(\mathfrak{x}) \in \mathfrak{z}$ whose ζ-function is non-trivial on the surface on which $c(\mathfrak{n})$ lies. To this effect we choose for each $\mathfrak{p}\in S$ some function $f_{\mathfrak{p}}(\mathfrak{x}_{\mathfrak{p}}) \in \mathfrak{z}_{\mathfrak{p}}$ whose (local) ζ-function is non-trivial on the surface on which $c_{\mathfrak{p}}$ lies (for instance select $f_{\mathfrak{p}}$ to be the function used to compute $\rho_{\mathfrak{p}}(c_{\mathfrak{p}} \mid 1_{\mathfrak{p}}^{s})$ in § 2.5). For $\mathfrak{p} \notin S$, we let $f_{\mathfrak{p}}(\mathfrak{x}_{\mathfrak{p}})$ be the characteristic function of the set $\mathfrak{o}_{\mathfrak{p}}$. We then put

$$f(\mathfrak{x}) = \prod_{\mathfrak{p}} f_{\mathfrak{p}}(\mathfrak{x}_{\mathfrak{p}}).$$

(We will show in the course of our computations that this $f(\mathfrak{x})$ is in the class $\mathfrak{Z}$.)

<u>Their Fourier Transforms:</u> According to lemma 3.3.3,

$$\hat{f}(\mathfrak{x}) = \prod_{\mathfrak{p}} \hat{f}_{\mathfrak{p}}(\mathfrak{x}_{\mathfrak{p}}),$$

and moreover, $f \in \mathfrak{B}_1(V)$, i.e. f satisfies axiom $\mathfrak{Z}_1)$. Notice that $\hat{f}(\mathfrak{x})$ is the same type of function as $f(\mathfrak{x})$, except for the fact that at those $\mathfrak{p} \notin S$ where $\mathfrak{d}_{\mathfrak{p}} \neq \mathfrak{o}_{\mathfrak{p}}$, $\hat{f}_{\mathfrak{p}}(\mathfrak{x}_{\mathfrak{p}})$ equals $N\mathfrak{d}_{\mathfrak{p}}^{-\frac{1}{2}}$ times the characteristic function of $\mathfrak{d}_{\mathfrak{p}}^{-1}$, rather than the characteristic function of $\mathfrak{o}_{\mathfrak{p}}$.

<u>The ζ-Functions:</u> Since $|f(\mathfrak{a})||\mathfrak{a}|^{\sigma} = \prod_{\mathfrak{p}} |f_{\mathfrak{p}}(\mathfrak{a}_{\mathfrak{p}})||\mathfrak{a}|_{\mathfrak{p}}^{\sigma}$ is a product of local functions, almost all of which are 1 on $\mathfrak{U}_{\mathfrak{p}}$, we may use theorem 3.3.1 to check the summability of $|f(\mathfrak{a})||\mathfrak{a}|^{\sigma}$ for $\sigma > 1$. A simple computation shows, for $\mathfrak{p} \notin S$,

$$\int_{k_{\mathfrak{p}}^{\times}} |f_{\mathfrak{p}}(\mathfrak{a}_{\mathfrak{p}})||\mathfrak{a}_{\mathfrak{p}}|_{\mathfrak{p}}^{\sigma}\, d\mathfrak{a}_{\mathfrak{p}} = \frac{N\mathfrak{d}_{\mathfrak{p}}^{-\frac{1}{2}}}{1 - N\mathfrak{p}^{-\sigma}}$$

The summability follows therefore from the well-known fact that the product

$$\prod_{\mathfrak{p} \notin S_{\infty}} \frac{1}{1 - N\mathfrak{p}^{-\sigma}}$$

is convergent for $\sigma > 1$. Well-known as this fact is it should be stressed that it is a keystone of the whole theory. The existence of our ζ-functions, just as that of the classical functions, depends on it. It is proved by descending directly to the basic field of rational numbers (see for example Landau, <u>Algebraische Zahlen</u>, 2nd edition, pages 55 and 56.). Because $\hat{f}(\mathfrak{x})$ is the same type of function as $f(\mathfrak{x})$, we see that $|\hat{f}(\mathfrak{a})||\mathfrak{a}|^{\sigma}$ is also summable for $\sigma > 1$. Therefore $f(\mathfrak{x})$ satisfies axiom $\mathfrak{Z}_3)$.

Having established the summability, we can also use theorem 3.3.1 to express the ζ-function as a product of local ζ-functions. Namely,

$$\zeta(f,c) = \prod_{\mathfrak{p}} \zeta_{\mathfrak{p}}(f_{\mathfrak{p}}, c_{\mathfrak{p}})$$

for any quasi-character $c = \prod_{\mathfrak{p}} c_{\mathfrak{p}}$ of exponent greater than 1. If c now denotes our special character, $c(\mathfrak{a}) = \prod_{\mathfrak{p}} c_{\mathfrak{p}}(\mathfrak{a}_{\mathfrak{p}}) = \prod_{\mathfrak{p}\in S} c_{\mathfrak{p}}(\mathfrak{a}_{\mathfrak{p}}) \cdot \chi(\varphi_S(\mathfrak{a}))$, we can compute explicitly the local factors of $\zeta(f, c|\,|^s)$ for $\mathfrak{p} \notin S$.

Indeed,

$$\begin{aligned}\zeta_{\mathfrak{p}}(f_{\mathfrak{p}}, c_{\mathfrak{p}}|\,|_{\mathfrak{p}}^s) &= \int_{\mathfrak{o}_{\mathfrak{p}}} c_{\mathfrak{p}}(\mathfrak{a}_{\mathfrak{p}})|\mathfrak{a}_{\mathfrak{p}}|_{\mathfrak{p}}^s \, d^*\mathfrak{a}_{\mathfrak{p}} \\ &= \sum_{\nu=0}^{\infty} \chi(\mathfrak{p}^{\nu}) N\mathfrak{p}^{-\nu s} \cdot N\mathfrak{d}_{\mathfrak{p}}^{-\frac{1}{2}} \\ &= \frac{N\mathfrak{d}_{\mathfrak{p}}^{-\frac{1}{2}}}{1-\chi(\mathfrak{p})N\mathfrak{p}^{-s}}\end{aligned}$$

because, for $\mathfrak{p} \notin S$, $c_{\mathfrak{p}}(\mathfrak{a}_{\mathfrak{p}}) = \chi(\mathfrak{p}^{\operatorname{ord}_{\mathfrak{p}} \mathfrak{a}_{\mathfrak{p}}})$. If therefore we introduce the classical ζ-function $\zeta(s,\chi)$, defined for $\operatorname{Re}(s) > 1$ by the Euler product

$$\zeta(s,\chi) = \prod_{\mathfrak{p}\notin S} \frac{1}{1-\chi(\mathfrak{p})N\mathfrak{p}^{-s}} ,$$

we can write

$$\zeta(f, c|\,|^s) = \prod_{\mathfrak{p}\in S} \zeta_{\mathfrak{p}}(f_{\mathfrak{p}}, c_{\mathfrak{p}}|\,|_{\mathfrak{p}}^s) \cdot \prod_{\mathfrak{p}\notin S} N\mathfrak{d}_{\mathfrak{p}}^{-\frac{1}{2}} \cdot \zeta(s,\chi) .$$

We see that our $\zeta(f, c|\,|^s)$ is, essentially, the classical function $\zeta(s,\chi)$. It may be remarked here that we could have obtained directly the additive expression for $\zeta(s,\chi)$,

$$\zeta(s,\chi) = \sum_{\substack{\mathfrak{a} \text{ integral ideal} \\ \text{prime to } S}} \frac{\chi(\mathfrak{a})}{N\mathfrak{a}^s}$$

had we computed $\zeta(f, c|\,|^s)$ by breaking up I into the cosets of I_S, integrating over each coset, and summing the results, rather than by using theorem 3.3.1 to express the ζ-function integral as a product of local integrals.

Treating the ζ-function of $\hat{f}$ in the same way we find

$$\zeta(\hat{f}, \widehat{c|\,|^s}) = \prod_{\mathfrak{p}\in S} \zeta(\hat{f}_{\mathfrak{p}}, \widehat{c_{\mathfrak{p}}|\,|_{\mathfrak{p}}^s}) \cdot \prod_{\mathfrak{p}\notin S} \chi(\mathfrak{d}_{\mathfrak{p}}) N\mathfrak{d}_{\mathfrak{p}}^{-s} \cdot \zeta(1-s,\chi^{-1}),$$

for $\operatorname{Re}(s) < 0$.

Before discussing the resulting analytic continuation and functional equation for $\zeta(s,\chi)$, we should set our minds completely at rest by checking that our $f(\mathfrak{y})$ satisfies axiom $\mathfrak{Z}_2)$, that is, that the sum

$$\sum_{\xi\in k} f(\mathfrak{M}(\mathfrak{y}+\xi))$$

is uniformly convergent for $\mathfrak{M}$ in a compact subset of I and $\mathfrak{y}\in D$. We can do this easily under the assumption that, for $\mathfrak{y}\in S$, the local functions $f_{\mathfrak{y}}$ we chose in constructing f do not differ too much from the standard local functions which we wrote down in § 2.5. Namely, we assume for discrete $\mathfrak{y}\in S$, that $f_{\mathfrak{y}}$ vanishes outside a compact; and for archimedean $\mathfrak{y}$, that $f_{\mathfrak{y}}(\mathfrak{y}_{\mathfrak{y}})$ goes exponentially to zero as $\mathfrak{y}_{\mathfrak{y}}$ tends to infinity. Under these assumptions one sees first that there is an ideal, $\mathfrak{O}\mathfrak{T}$, of k such that $f(\mathfrak{M}(\mathfrak{y}+\xi)) = 0$ if $\xi\notin\mathfrak{O}\mathfrak{T}$, for all $\mathfrak{M}$ in the compact and $\mathfrak{y}$ in D. The sum may then be viewed as a sum over a lattice in the n-dimensional space which is the infinite part of V, of the values of a function which goes exponentially to zero with the distance from the origin. The lattice depends on $\mathfrak{M}$ and $\mathfrak{y}$, to be sure, but the restriction of $\mathfrak{M}$ to a compact means that a certain fixed small cube will always fit into the fundamental parallelotope of the lattice. The uniform convergence of the sum is then obvious.

Analytic Continuation and Functional Equation for $\zeta(s,\chi)$: The analytic continuation which we have established for our ζ-functions, both in the large and locally, now gives directly the analytic continuation of $\zeta(s,\chi)$ into the whole plane. Our functional equations

$$\zeta(\hat{f},\hat{c})=\zeta(f,c) \quad\text{and}\quad \zeta_{\mathfrak{y}}(f_{\mathfrak{y}},c_{\mathfrak{y}}) = \rho_{\mathfrak{y}}(c_{\mathfrak{y}})\,\zeta_{\mathfrak{y}}(\hat{f}_{\mathfrak{y}},\hat{c}_{\mathfrak{y}})$$

yield for $\zeta(s,\chi)$ the functional equation

$$\zeta(1-s,\chi^{-1}) = \prod_{\mathfrak{y}\in S}\rho_{\mathfrak{y}}(\tilde{c}_{\mathfrak{y}}|\ |_{\mathfrak{y}}^{s+it_{\mathfrak{y}}}) \cdot \prod_{\mathfrak{y}\notin S}\chi^{-1}(\vartheta_{\mathfrak{y}})\,N\vartheta_{\mathfrak{y}}^{s-\frac{1}{2}}\cdot\zeta(s,\chi).$$

(4.24)

The explicit expressions for the local functions $\rho_{\mathfrak{p}}$ are tabulated in §2.5. The meaning of the $\tilde{c}_{\mathfrak{p}}$ and $t_{\mathfrak{p}}$, and their relationship to the ideal character χ, is discussed in the first paragraph of this section.

These ideal characters, χ, which we have constructed out of idèle characters, are exactly the characters which Hecke introduced in order to define his "new type of ζ-function". $\zeta(s,\chi)$ is that ζ-function; and the functional equation we have just written down is the functional equation Hecke proved for it.

References.

1. Hecke, E. Über eine neue Art von Zetafunctionen und ihre Beziehungen zur Verteilung der Primzahlen. Math. Zeitschr. 1 (1918) and 4 (1920).

2. Chevalley, C. La Théorie du Corps de Classes, Ann. of Math. vol. 41 (1940) pp. 394-418.

3. Artin, E. and Whaples, G. Axiomatic Characterization of Fields by the Product Formula for Valuations. Bull. Amer. Math. Soc., vol. 51, No. 7, (1945) pp. 469-492.

4. Matchett, M. Thesis, Indiana University (1946), unpublished.

5. Cartan, H. and Godemont, R. Théorie de la dualité et analyse harmonique dans les groups abéliens localement compacts. Ann. Sci. Ecole Norm. Sup. (3), 64 (1947), pp. 79-99.