Last time: $f \in S(\mathbb{A})$, proved:

(adelic) Poisson Summation: $\displaystyle\sum_{q \in \mathbb{Q}} f(qt) = \frac{1}{|t|_{\mathbb{A}}} \sum_{r \in \mathbb{Q}} \hat{f}\left(\frac{r}{t}\right)$

"Dilated" $\forall t \in \mathbb{A}^\times$

$$\vartheta_f(t) := \sum_{q \in \mathbb{Q}^\times} f(qt) \quad \text{on} \quad \mathbb{Q}^\times \backslash \mathbb{A}^\times \left(= GL(1,\mathbb{Q}) \backslash GL(1,\mathbb{A}) \right).$$

Mellin transform: $\operatorname{Re} s > 1$.

$$\tilde{\vartheta}_f(s) = \int_{\mathbb{Q}^\times \backslash \mathbb{A}^\times} \vartheta_f(t)\, |t|^s\, d^\times t, \qquad \text{a.a. } f_p = \mathbb{1}_{\mathbb{Z}_p}.$$

(unfold)

$$= \int_{\mathbb{A}^\times} f(t)\, |t|^s\, d^\times t = \prod_{p \le \infty} \underbrace{\int_{\mathbb{Q}_p^\times} f_p(t_p)\, |t_p|_p^s\, d^\times t_p}_{I(p)}$$

Could choose $f\begin{pmatrix}\mathbb{A} \\ \psi \\ x\end{pmatrix} = e^{-\pi x_\infty^2} \cdot \prod_p \mathbb{1}_{\mathbb{Z}_p}(x_p) \in S(\mathbb{A})$,

"$\mathbb{1}_{\mathbb{Z}_p}$ is the p-adic Gaussian".

Then $I(\infty) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)$, $I(p) = \frac{1}{1-\frac{1}{p^s}}$.

$$\tilde{\vartheta}_f(s) \overset{\mathrm{Re}\, s>1}{=} \pi^{-s/2}\Gamma\left(\frac{s}{2}\right) \underbrace{\prod_p \left(1-\frac{1}{p^s}\right)^{-1}}_{= \zeta(s)} \quad \left(\overset{\text{Unique factorization}}{=} \sum_{n\geq 1} \frac{1}{n^s}\right).$$

$\frac{1}{1-\lambda} = 1+\lambda+\lambda^2+\ldots$

Back to general $f \in S(\mathbb{A})$.

$$\tilde{\vartheta}_f(s) = \int_{\mathbb{Q}^\times \backslash \mathbb{A}^\times} \vartheta_f(t)\, |t|^s\, d^\times t.$$

Recall $D^\times = (0,\infty) \times \prod \mathbb{Z}_p^\times$

$$= \int_{t \in D^\times} \vartheta_f(t)\, |t|^s\, d^\times t + \int_{|t|_{\mathbb{A}} \leq 1} \ldots$$

$|t_\infty|_\infty \prod_p |t_p|_p = |t|_{\mathbb{A}} \geq 1$

$$= \int_{\substack{t \in D^\times \\ |t|_{\mathbb{A}} \geq 1}} \Big[\underbrace{\sum_{q \in \mathbb{Q}^\times} f(qt)}_{= \sum_{n \in \mathbb{Z}\setminus\{0\}} f\left(\frac{n}{N}t\right)}\Big] |t|_{\mathbb{A}}^s\, d^\times t + I_{\leq}.$$

entire function of $s \in \mathbb{C}$.

Classically: $\displaystyle\int_1^\infty \sum_{n\geq 1} |f(n\cdot t)|\,|t|^{\operatorname{Re} s}\,\frac{dt}{t}$.

$$I_\leq = \int_{\substack{t\in \mathbb{A}^\times \\ |t|\leq 1}} \vartheta_f(t)\,|t|^s\,d^\times t$$

$$\vartheta_f(t) = \sum_{q\in\mathbb{Q}^\times} f(qt) = \underbrace{\frac{1}{|t|}\sum_{r\in\mathbb{Q}^\times}\hat{f}\left(\frac{r}{t}\right)}_{\frac{1}{|t|}\vartheta_{\hat f}\left(\frac{1}{t}\right)} + \frac{1}{|t|}\hat{f}(0) - f(0)$$

$$= \underbrace{\int_{\substack{t\in\mathbb{A}^\times \\ |t|\leq 1}} \left[\frac{1}{|t|}\vartheta_{\hat f}\left(\frac{1}{t}\right)\right]|t|^s\,d^\times t}_{t\mapsto 1/t} + \int_{\substack{t\in\mathbb{A}^\times \\ |t|\leq 1}} \frac{\hat{f}(0)}{|t|}|t|^s\,d^\times t - \int_{\substack{t\in\mathbb{A}^\times \\ |t|\leq 1}} f(0)\,|t|^s\,d^\times t.$$

$$= \int_{\substack{t\in\mathbb{A}^\times \\ |t|\geq 1}} \vartheta_{\hat f}(t)\,|t|^{1-s}\,d^\times t.$$ entire in $s \in \mathbb{C}$.

$$\hat{f}(0) \cdot \int_0^1 t_\infty^{s-1} \frac{dt_\infty}{t_\infty} \cdot \prod_p \underbrace{\int_{\mathbb{Z}_p^\times} |t_p|^{\cancel{s}\,1} d^\times t_p}_{1}$$

$$= \frac{\hat{f}(0)}{s-1}$$

$$- \frac{f(0)}{s}.$$

Summarize:

$$\vartheta_f(s) \overset{\operatorname{Re} s > 1}{=} \prod_{p \in S} (\quad) \cdot \prod_{p \notin S} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

$$\| \quad \underbrace{\int_{\substack{t \in \mathbb{A}^\times \\ |t| \geq 1}} \left[\vartheta_f(t)\,|t|^s + \vartheta_{\hat{f}}(t)\,|t|^{1-s}\right] d^\times t}_{\text{entire.}} - \frac{\hat{f}(0)}{1-s} - \frac{f(0)}{s},$$

Poles at $s = 0, 1$.

Meromorphic cont of ζ to all $s \in \mathbb{C}$. ← (Riemann).

Applications: Prime Number Theorem (Gauss's Conj)

$$\#\{p < x\} = \overbrace{\int_2^x \frac{dt}{\log t}}^{= \mathrm{Li}(x)} + O\left(x^{.99}\right)$$

$x^{1/2+\varepsilon}$ ← Riemann hyp.

Say we want to know $\#p < 10^{100}$ $\sim \frac{x}{\log x}$ (crude!). Ans: $\frac{10^{100}}{100} + O(10^{50})$ on RH

Without counting primes, first 50 digits of answer would be accurate, I.e. first $1/2$ of digits. Know today: 0% digits.

Not hard: $O\left(x e^{-c\sqrt{\log x}}\right)$ ← better than $O\left(\frac{x}{(\log x)^{1000}}\right)$.

Compare: $O\left(x^1 \cdot \underbrace{x^{-.01}}_{e^{-0.1 \log x}}\right)$.

Compare: $\underline{\text{main}}\left(x \cdot \underbrace{(\log x)^{-1}}_{e^{-\log\log x}}\right)$

Other classical applications:

Are there infinitely many primes $\equiv 17 \bmod 23$?. (Euclid)

17, 40, 63, 86, 109, 132, —

Special cases of $p \equiv a \bmod q$ $(a,q)=1$,

were known before Dirichlet 1837

(following Euler): $\sum_{p \equiv a(q)} \frac{1}{p} = \infty$.

Key objects discovered: Dirichlet Char:

$(\mathbb{Z}/N\mathbb{Z})^\times = \{ a \bmod N : (a,N)=1 \}$.

$= \{ a \bmod N : \exists\, b,\ a \cdot b \equiv 1\ (N) \}$.

multiplicative Characters $\chi : \mathbb{Z}/N^\times \to \mathbb{C}^\times$?.

$N=5$, $\mathbb{Z}/5^\times = 1, 2, 3, \overset{-1}{4}$.

structure of $(\mathbb{Z}/p)^\times \simeq \mathbb{Z}/p-1$ (Gauss primitive next time)

$(\mathbb{Z}/p)^\times$ always has a generator $= \langle a \rangle$.

Is 2 a generator for p i.o.?

(mod 5) $a=2$, $a^2 = 4$, $a^3 = 3$, $a^4 \equiv 1$.

FLT. Fermat $a^{p-1} \equiv 1 \mod p$, $(a,p)=1$

Know: $2^4 \equiv 1$, $\chi(2)^4 = \chi(2^4) = \chi(1) = 1$

a	1	2	3	4
χ_1	1	1	1	1
χ_2	1	-1	-1	1
χ_3	1	i	$-i$	-1
χ_4	1	$-i$	i	-1

← trivial

imprimitive

let $\widehat{(\mathbb{Z}/5)^\times}$ = group of char. $= \{\chi_1, \dots, \chi_4\}$. $= \langle \chi_3 \rangle$.

$N=12$. $(\mathbb{Z}/12)^\times = 1, 5, 7, 11 = \mathbb{Z}/2 \times \mathbb{Z}/2$.

$5^2 = 1$ $7^2 = 1$ $(-1)^2 = 1$

Char table:

$G = \mathbb{Z}/2 \times \mathbb{Z}/2.$

a	1	5	7	11	
χ_1	1	1	1	1	← trivial character
χ_2	1	1	-1	-1	← primitive
→ χ_3	1	-1	1	-1	← periodic mod $6 \mid 12$. imprimitive
→ χ_4	1	-1	-1	1	← primitive.

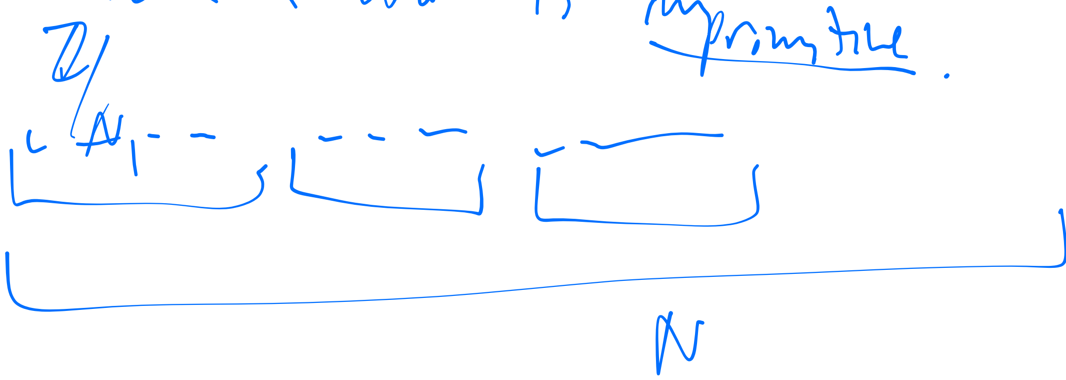
Funny thing to do: extend $\chi : (\mathbb{Z}/N)^\times \to \mathbb{C}^\times$,

to $\chi : \mathbb{Z}/N \to \mathbb{C}$, via $\chi(a) = 0$.
$(a,N) > 1$

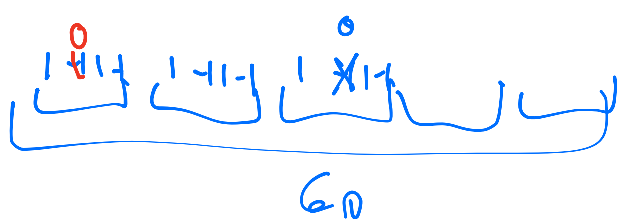
Def: If $N_1 \mid N$, χ_1 is a char mod N_1,

def induced character $\chi(a) = \begin{cases} \chi_1(a) & \\ 0 & (a,N)>1 \end{cases}$

Such a char is imprimitive.



Ex: $N_1 = 12$, $N = 60$,



Def χ is primitive if $\nexists\, N' | N$, χ_1 s.t. χ induced from χ_1.

Given χ on $\mathbb{Z}/N^{\times}$, $\mathbb{Z}/N$, extend again to χ on $\mathbb{N} \rightsquigarrow \mathbb{Z}$. (compose with mod N reduction).

"Twisted" Poisson summation $f \in \mathcal{S}(\mathbb{R})$

$$\sum_{n \in \mathbb{Z}} f(n)\chi(n) \overset{?}{=} ?$$

$\to$ the most extension $\chi : \mathbb{R} \to \mathbb{C}$,

Def: Dirichlet L-function, $L(s) = \sum_{n \geq 1} \frac{\chi(n)}{n^s}$.

$$\| \quad L_\chi(s) = L(s,\chi) = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}$$

e.g. χ mod 5 : $1, i, -i, -1$,

$$L(s) = \frac{1}{1^s} + \frac{i}{2^s} - \frac{i}{3^s} + \frac{-1}{4^s} + \frac{0}{5^s} + \frac{1}{6^s} + \frac{i}{7^s} - \frac{i}{8^s} \cdots$$

$\chi(n \cdot m) = \chi(n)\chi(m)$ mod N.