Last time: Classically, Poisson Summation

$f \in S(\mathbb{R})$, $t > 0$.

$$\sum_{n\in\mathbb{Z}} f\left(\frac{n}{t}\right) = \frac{1}{t}\sum_{m\in\mathbb{Z}} \hat{f}\left(\frac{m}{t}\right)$$

$$\Rightarrow \quad \vartheta_f(t) = \sum_{n \geq 1} f(nt), \quad \tilde{\vartheta}_f(s) = \int_0^\infty \vartheta_f(t)\, t^s \frac{dt}{t}$$

$$\xrightarrow{\mathrm{Re}\, s > 1} = \sum_{n\geq 1} n^{-s} \cdot \tilde{f}(s).$$

Now take $f \in S(\mathbb{A})$. $f: \mathbb{A} \to \mathbb{C}$. $f = \prod f_p$.

- $f_\infty \in S(\mathbb{R})$,
- f_p loc const, cpt supp,
- a.a. p. $f_p = \mathbb{1}_{\mathbb{Z}_p}$.

$$\hat{f}(\xi) = \int_{\mathbb{A}} f(x)\, e_{\mathbb{A}}(x \cdot \xi)\, dx \in S(\mathbb{A}).$$

Thm (Poisson summation):
$$\sum_{q\in\mathbb{Q}} f(q) = \sum_{r\in\mathbb{Q}} \hat{f}(r)$$

Pf: Automorphize / average / periodize:

Let $F\begin{pmatrix}\mathbb{A}\\ \cup \\ X\end{pmatrix} := \sum_{q \in \mathbb{Q}} f(x+q)$.

Converges? (Uniformly on cpta?). Recall $f \in S(\mathbb{A})$,

$\exists S_0$ s.t. $\forall p \notin S_0$, $f_p = \mathbb{1}_{\mathbb{Z}_p}$.

$\exists S_1$ s.t. $\forall p \notin S_1$, $\underline{x_p \in \mathbb{Z}_p}$. (True unif. in cpt $K \ni x$),

Let $S := S_0 \cup S_1$. When is $f(x+q) \neq 0$?

$= \underbrace{\prod_p f_p(x_p + (q : \mathbb{Q}_p))}$.

For $p \notin S_1$, $f_p(x_p + q) \neq 0$ if only if $(q : \mathbb{Q}_p) \in \mathbb{Z}_p$.

$\Leftrightarrow$ No p's in denominator of q.

So only q's that contribute to $\sum_{q \in \mathbb{Q}}$

are of the form $\dfrac{n}{\prod_{p \in S} p^{e_p}}$. $\leftarrow$ Set of such not discrete in $\mathbb{R}$.

Even for $p \in S$, f_p cptly supp. $\left(V_\infty \times \prod_{p \in S_0} p^{b_p}\mathbb{Z}_p \times \prod \mathbb{Z}_p \right)$

$y_p \in p^{a}\mathbb{Z}_p$. uniform over $K \ni x$. $\forall y \in K$

$x = (x_\infty, \underbrace{x_2, x_3,}_{p^{a_p(x)}\mathbb{Z}_p} \underbrace{\cdots}_{\mathbb{Z}_p}) \in \mathbb{A}$.

$$\boxed{\operatorname{Supp} f \subset \mathbb{R} \times \prod_{p \in S_0} p^{b_p}\mathbb{Z}_p \times \prod_{p \notin S_0} \mathbb{Z}_p.}$$

$\underbrace{K = \underset{x_\infty}{\underset{\Psi}{U_\infty}} \times \prod_{p \in S,} p^{a_p(x)}\mathbb{Z}_p \times \prod_{p \notin S,} \mathbb{Z}_p}$ $\ni x$.

Let $c_p = \min(a_p, b_p)$ (neg integers).

If q's s.t. $f(y+q) \neq 0$, claim

$$q = \frac{n}{\prod_{p \in S} p^{e_p}}, \quad e_p < -c_p.$$

Let $N = \prod_{p \in S} p^{-c_p} \in \mathbb{N}$. Then

$$\sum_{q\in\mathbb{Q}} |f(x+q)| \le \sum_{n\in\mathbb{Z}} \Big|f\big(x+\tfrac{n}{N}\big)\Big|.$$
$$\Big|f\big(x+\tfrac{n}{N}\big)\Big| = \prod_{p\le\infty} \Big|f_p\big(x_p+\tfrac{n}{N}\big)\Big|$$

$$\le C_f \cdot \sum_{n\in\mathbb{Z}} \Big|f_\infty\big(x+\tfrac{n}{N}\big)\Big|$$

abs conv unif on cpta.

$$F(x) = \sum_{q\in\mathbb{Q}} f(x+q) \quad \text{on } \mathbb{Q}\backslash\mathbb{A}.$$

D fund dom $= (0,1)\times\prod_p \mathbb{Z}_p$.

$$\bigcup_{q\in\mathbb{Q}} (q+\bar{D}) = \mathbb{A}.$$

Step 2:

$$F(x) = \sum \hat{F}(r)\, e_{\mathbb{A}}(-rx).$$

$r \in \mathbb{Q}$

Fourier inversion on $\mathbb{A}/\mathbb{Q}$.

To make of this sum, need to compute:

$$\hat{F}(r) = \int_{\mathbb{Q}\backslash\mathbb{A}} F(x)\, e_{\mathbb{A}}(rx)\, dx.$$

fix fund dom.

$$= \int_D \sum_{q \in \mathbb{Q}} f(x+q)\, e_{\mathbb{A}}(rx)\, dx,$$

abs conv

$$= \sum_{q \in \mathbb{Q}} \int_D f(x+q)\, e_{\mathbb{A}}(rx)\, dx.$$

$y = x+q, \quad dy = dx.$

invariant Haar

$$= \sum_{q \in \mathbb{Q}} \int_{D+q} f(y)\, \underbrace{e_{\mathbb{A}}(r(y-q))}\, dy.$$

char $= e_{\mathbb{A}}(ry) \cdot \cancel{e_{\mathbb{A}}(-rq)}$!

triv on $\mathbb{Q}$.

$$= \underbrace{\sum_{q \in \Omega} \int_{D+q}}\ \underbrace{f(y)\, e_{\mathbb{A}}(ry)\, dy}_{\text{no } q \text{ dep.}}$$

$$= \int_{\mathbb{A}} f(y)\, e_{\mathbb{A}}(ry)\, dy = \hat{f}(r).$$

$$F(x) = \sum_{r \in \Omega} \hat{F}(r)\, e_{\mathbb{A}}(-rx)$$

$$\parallel$$

$$\sum_{q \in \Omega} f(x+q) = \underbrace{\sum_{r \in \Omega} \hat{f}(r)\, e_{\mathbb{A}}(-rx)}_{\text{converges absolutely}}.$$

Set $x=0$.

$$\sum_{q \in \Omega} f(q) = F(0) = \sum_{r \in \Omega} \hat{f}(r) \quad \text{■}.$$

Fix $t \in \mathbb{A}^\times$. Let $f_t(x) := f(x \cdot t)$.

Still Schwartz? a.e. t, a.a. p, $f_p(x_p \cdot t_p) = \mathbb{1}_{\mathbb{Z}_p}$?

(adelic)

Yes.

$$\Rightarrow \boxed{\sum_{q \in \mathbb{Q}} f(q \cdot t) = \frac{1}{|t|} \sum_{r \in \mathbb{Q}} \hat{f}\left(\frac{r}{t}\right)}$$

where $\hat{f}_t(\xi) = \int_{\mathbb{A}} f(x \cdot t)\, e_{\mathbb{A}}(x \cdot \xi)\, dx.$

$y = x \cdot t$. $\quad dx = \prod_{p \leq \infty} dx_p$

$dy = dx \cdot |t|_{\mathbb{A}}$. $\quad c_p \cdot |x_p|_p$

$$= \int_{\mathbb{A}} f(y)\, e_{\mathbb{A}}\left(\frac{y}{t} \xi\right) \frac{dy}{|t|_{\mathbb{A}}}.$$

$$= \frac{1}{|t|_{\mathbb{A}}} \cdot \hat{f}\left(\frac{\xi}{t}\right)$$

$$\sum_{q\in\mathbb{Q}} f(q\cdot t) = \frac{1}{|t|}\sum_{r\in\mathbb{Q}} \hat{f}\left(\frac{r}{t}\right)$$

Let $\vartheta_f(t) := \sum_{q\in\mathbb{Q}^\times} f(q\cdot t)$ on $\mathbb{A}^\times / \mathbb{Q}^\times$.

$$= \frac{1}{|t|}\vartheta_{\hat{f}}\left(\frac{1}{t}\right) + \frac{\hat{f}(0)}{|t|} - f(0).$$

Try: $\tilde{\vartheta}_f(s) := \cancel{\int_{\mathbb{A}^\times} \vartheta_f(t)\cdot |t|_{\mathbb{A}}^s \, d^\times t}$

$= \infty.$

Recall: $\vartheta_f(t) = \sum f(q\cdot t)$ $\leftarrow$ no automorphy!

$\vartheta_f(t) = \sum_{n \geq 1} f(nt)$. $\mathbb{N}$ not a group.

Adelically, $\vartheta_f(t)$ is def on $\mathbb{Q}^\times \backslash \mathbb{A}^\times$,

Let $\tilde{\vartheta}_f(s) := \int_{\mathbb{Q}^\times \backslash \mathbb{A}^\times} \vartheta_f(t)\, |t|_{\mathbb{A}}^s \, d^\times t$.

$D^\times = (0,\infty) \times \prod_{p<\infty} \mathbb{Z}_p^\times$ fund dom for $\mathbb{Q}^\times \backslash \mathbb{A}^\times$.

(Re $s>1$), abs conv

$$= \sum_{q \in \mathbb{Q}^\times} \int_{D^\times} f(\underbrace{q \cdot t})\, |t|_{\mathbb{A}}^s \, d^\times t$$

$u = q \cdot t$. $d^\times u = d^\times t$.

Haar

$$= \sum_{q \in \mathbb{Q}^\times} \int_{q \cdot D^\times} f(u) \left|\frac{u}{q}\right|_{\mathbb{A}}^s d^\times u .$$

mult char

$$= \sum_{q \in Q^\times} \int_{q \cdot D^\times} f(u) \, |u|^s \, \cancel{|q|_A^{-s}} \, d^\times u \to 1.$$

no q's.

unfold

$$= \boxed{\int_{A^\times} f(u) \, |u|^s \, d^\times u}.$$

$$= \prod_{p \leq \infty} \int_{Q_p^\times} f_p(u_p) \, |u_p|_p^s \, d^\times u_p.$$

$f_p = 1_{Z_p}$.

$$= \tilde{f}_\infty(s) \cdot \prod_{p \in S} (\quad) \prod_{p \notin S}$$

$$\int |u|^s \, du_p$$

$$\int_{\mathbb{Z}_p \setminus \{0\}} \ldots |u_p|_p \ldots \frac{\ldots}{|u_p|_p \cdot \left(1-\frac{1}{p}\right)}$$

$$= \sum_{n \geq 0} \int_{p^n \cdot \mathbb{Z}_p^\times} p^{-ns} \, d^\times u_p = \frac{1}{1-p^{-s}}.$$

$$\int_{\mathbb{A}^\times} f(u) \, |u|^s \, d^\times u = \prod_{p \in S} \zeta_p(s) \cdot \underbrace{\prod_{p \notin S} \frac{1}{1-p^{-s}}}_{\zeta^S(s)}.$$

Euler product comes for free when working $\mathbb{A}^{(\times)}$.

<u>Left:</u> FE, anal, $s \not> 2$ cont. ...