<u>Last time:</u> Strong Approx: a fund dom for action of $\mathbb{Q} \curvearrowright \mathbb{A}$ is: $(0,1) \times \prod_p \mathbb{Z}_p$.

<u>"Weaker" Approx:</u> Given S finite, $c_p \in \mathbb{Z}_p$ $(p \in S)$, & $\varepsilon > 0$, $\exists\, n \in \mathbb{Z}$: $|n - c_p|_p < \varepsilon$. <u>Pf:</u> CRT.

<u>Weak Approx (WA):</u> Given S finite, $c_p \in \mathbb{Q}_p$ $(p \in S)$, & $\varepsilon > 0$, $\exists\, q \in \mathbb{Q}$: $|q - c_p|_p < \varepsilon$. $\iff$ $|q \cdot p^{e_p} - c_p \cdot p^{e_p}|_p < \varepsilon \cdot p^{e_p}$.

<u>Pf:</u> $\forall p \in S$, $\exists\, e_p \geq 0$ s.t. $c_p \cdot p^{e_p} \in \mathbb{Z}_p$.

Let $N = \prod_{p \in S} p^{e_p}$, $N_p = \frac{N}{p^{e_p}} \in \mathbb{Z}_p^{\times}$.

Apply <u>WerA</u> to $\Big(\{\underbrace{c_p \cdot p^{e_p} \cdot N_p}_{\in \mathbb{Z}_p}\}_{p \in S},\ \underbrace{\min(\varepsilon \cdot p^{-e_p})}\Big)$

Get: n s.t.

$$|n - c_p \cdot p^{e_p} \cdot N_p|_p < \varepsilon \cdot p^{-e_p}.$$

Use $q = \frac{n}{N}$. $\iff \left|\frac{n \cdot p^{-e_p}}{N_p} - c_p \cdot N_p\right|_p < \varepsilon.$ $\underset{=}{q.}$

& $|N_p^{-1}|_p = 1$.

Ideles: $\mathbb{A}^\times = \{ x \in \mathbb{A} : \forall p, x_p \in \mathbb{Q}_p^\times, \text{ a.a. } p\ x_p \in \mathbb{Z}_p^\times \}$,

top stronger than <u>subspace</u>. gen $\prod_{p \in S} U_p \times \prod_{p \notin S} \mathbb{Z}_p^\times$,

$U_p \subset \mathbb{Q}_p^\times$ open

Claim: $\mathbb{Q}^\times \subset \mathbb{A}^\times$. pf: finitely many primes in denominator (into $\mathbb{Q}_p$) & numerator (into $\mathbb{Z}_p^\times$)

$\uparrow$ mult gp. eventually. $\boxed{G/\Gamma \text{ vs } \Gamma\backslash G}$

Action $\mathbb{Q}^\times \curvearrowright \mathbb{A}^\times$, $\overset{\mathbb{Q}^\times}{q} : \overset{\mathbb{A}^\times}{x} \mapsto x \cdot q = \underline{\underline{q \cdot x}}$.

Is this action discrete? I.e. is there an open set around $\underline{1} = ((1:\infty), (1:a_{\infty}), \ldots)$, with no other $\mathbb{Q}^\times$ pts in it? Let

$$U = \overset{\mathbb{R}^\times_{+}}{(0,\infty)} \times \prod_p \mathbb{Z}_p^\times .$$ Then if $\overset{\mathbb{Q}^\times}{q} \in U$,

then $(q : a_p) \in \mathbb{Z}_p^\times \Rightarrow q$ has no p's in numerator nor denom.

$\Rightarrow q = \pm 1$. $\Rightarrow q = 1$.

Thm: This U is a fund dom for $\mathbb{Q}^\times \subset \mathbb{A}^\times$,

pf: ② If x & $y \in U$ & $x = y \cdot \overset{\mathbb{Q}^\times}{q}$. Want: $q = 1$. (& $x = y$.)

Look at $x \cdot y^{-1} \in U \Rightarrow xy^{-1} = 1 \Rightarrow q = 1$.

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① Given $x = (x_\infty, x_2, \ldots) \in \mathbb{A}^\times$, $\exists!$ $q \in \mathbb{Q}$ s.t. $x \cdot q \in U$? There are fin many p s.t. $x_p \notin \mathbb{Z}_p^\times$.

For each such, $\exists$ $e_p \in \mathbb{Z}$ s.t. $p^{e_p} \cdot x_p \in \mathbb{Z}_p^\times$.

Let $q = \prod p^{e_p} \cdot \operatorname{sgn} x_\infty$. Then $x \cdot q \in \overline{U} = U$.

Recall: char $e_{\mathbb{A}}(x) = \prod_{p \le \infty} e_p(x_p)$, where

finite product a.a. $\in \mathbb{Z}_p$. $: \mathbb{A} \to \mathbb{C}^\times$.

for $p < \infty$, $e_p(x_p) = e^{2\pi i x_p}$. & $e_\infty(x_\infty) = e^{-2\pi i x_\infty}$.

Claim 1: $\forall q \in \mathbb{Q}$, $e_{\mathbb{A}}(q) = 1$. , $e_{\mathbb{A}}|_{\mathbb{Q}} \equiv 1$.

~~Claim 2:~~ Suffices to show $e_{\mathbb{A}}\left(\frac{1}{p^e}\right) = 1$.

Because: If $q = \frac{n}{p_1^{e_1} \cdots p_k^{e_k}}$; $(n, p_1 \cdots p_k) = 1$. Let $N = p_1^{e_1} \cdots p_k^{e_k}$,

$N_j = \frac{N}{p_j^{e_j}}$. Claim 3: $\exists\, b_j \in \mathbb{Z}$ s.t. $\frac{1}{N} = \frac{b_1}{p_1^{e_1}} + \cdots + \frac{b_k}{p_k^{e_k}}$.

Pf: Want: $1 = b_1 \cdot N_1 + b_2 \cdot N_2 + \cdots + b_k \cdot N_k$. (Bezout eqn).

$(N_1, N_2, \ldots, N_k) = 1$. Claim 4: Can be solved,

induction on Euclidean alg.

Ex. $N = 2^3 \cdot 3^2 \cdot 5$.

$N_1 = 3^2 \cdot 5$, $N_2 = 2^3 \cdot 5$, $N_3 = 2^3 \cdot 3^2 = 72$.

Step 1: Solve $d_1 \cdot N_1 + d_2 \cdot N_2 = 5$

$$N_1 - N_2 = 5.$$

$145 - 144 = 1$

Step induction:

$$29\,(\overbrace{N_1 - N_2}^{=\,?}) - 2N_3 = 1$$

$$\Rightarrow 29 \cdot N_1 - 29 N_2 - 2N_3 = 1 \quad \checkmark_{43}$$

Assume $\underline{e_{\mathbb{A}}\left(\frac{1}{p^e}\right) = 1}$. Then $\frac{n}{N} = \frac{nb_1}{p_1^{e_1}} + \dots + \frac{b_k n}{p_k^{e_k}}$

$$e_{\mathbb{A}}(q) = e_{\mathbb{A}}\left(\frac{n}{N}\right) = \underbrace{e_{\mathbb{A}}\left(\frac{nb_1}{p_1^{e_1}}\right)} \cdots e_{\mathbb{A}}\left(\frac{b_k \cdot n}{p_k^{e_k}}\right)$$

$= 1. \quad \checkmark_2.$

$$e_{\mathbb{A}}\left(\frac{1}{p^e}\right) = e_{\infty}\left(\frac{1}{p^e}\right) \cdot e_p\left(\frac{1}{p^e}\right)$$

$$= e^{-2\pi i \frac{1}{p^e}} \cdot e^{2\pi i \left(\frac{1}{p^e}\right)} = 1. \quad \checkmark_1$$

$$\frac{n}{p^e} = \frac{1}{p^e} + \dots + \frac{1}{p^e}$$

$$e_p\left(a_{-N} p^{-N} + \dots + a_{-1} p^{-1} + \underbrace{a_0 + a_1 p + \dots}\right) = e^{2\pi i \left(a_{-N} p^{-N} + \dots + a_{-1} p^{-1}\right)}.$$

I.e. $e_{\mathbb{A}}$ is $\mathbb{Q}$-periodic! $e_{\mathbb{A}}(x+q) = e_{\mathbb{A}}(x)$.

<u>Def.</u> $f : \mathbb{A} \to \mathbb{C}$ is <u>factorizable</u> if

$\exists\, f_p : \mathbb{Q}_p \to \mathbb{C}$ $(\forall p \leq \infty)$, a.a. p, $f_p\big|_{\mathbb{Z}_p} \equiv 1$,

s.t. $f(x) = \prod_{p \leq \infty} f_p(x_p)$ finite product $\forall$ fixed $x \in \mathbb{A}$.

<u>E.g.</u> $f = e_{\mathbb{A}}$.

<u>Def.</u> $F = \sum_{j=1}^{k} c_j f_j$ is called (adelic) <u>Schwartz</u> if it is (a finite linear combination) factorizable $\underbrace{f_j = \prod_{p \leq \infty} f_p}$, s.t.

f_∞ is Schwartz. & f_p loc const, cptly supp.

<u>Def.</u> Given (Bruhat) factorizable Schwartz $f = \prod_{p \leq \infty} f_p$, with f_p's

support in $C_p \subset \mathbb{Q}_p$ cpt, a.a. p, $C_p = \mathbb{Z}_p$, (i.e. a.a. p, $f_p = \mathbb{1}_{\mathbb{Z}_p}$), Then $\displaystyle\int_{\mathbb{A}} f \, d\mu := \prod_{p \leq \infty} \int_{\mathbb{Q}_p} f_p(x_p)\, dx_p.$ finite product.

Def, Given Schwartz (adelic, Bruhat, factorizable),

f, let $\hat{f}(\xi) := \displaystyle\int_{\mathbb{A}} \underbrace{f(x)\, e_{\mathbb{A}}(\xi \cdot x)}\, d\mu(x)$ $\xi \in \mathbb{A}$

$\boxed{\xi_p \text{ eventually in } \mathbb{Z}_p, \text{ so } \prod \text{ is finite}}$

Q: is $f \cdot e_{\mathbb{A}}$ Schwartz? $= \prod f_p \cdot e_p$. a.a. p, $f_p = \mathbb{1}_{\mathbb{Z}_p}$.

$\boxed{\Rightarrow \hat{f}_p = \mathbb{1}_{\mathbb{Z}_p} \Rightarrow \hat{f} \text{ is Schwartz.}}$

Recall: $\widehat{\mathbb{1}_{\{a + p^n \mathbb{Z}_p\}}}(\xi) = e^{2\pi i a \cdot \xi} \cdot \mathbb{1}_{\xi \in p^{-n}\mathbb{Z}_p}.$

Upcoming: Poisson Summation. $\mathbb{R}/\mathbb{Z}$.