Last time: Topology on $\mathbb{Q}_p$, is locally compact. ← noncompact

$\mathbb{Z}_p := \{x \in \mathbb{Q}_p : |x|_p \le 1\} = \{a_0 + a_1 p + a_2 p^2 + \dots\}$

Open balls of the form cylinder sets, $a \in \mathbb{Q}_p$, $n \in \mathbb{Z}$.

$a + p^n \mathbb{Z}_p$

Aside: In $\mathbb{R}$, $(\mathbb{R}, +)$ (Haar) measure invariant under action of $+$, i.e. fix $t \in \mathbb{R}$,

want: $d\mu(x)$ s.t. $\int_{\mathbb{R}} f(x+t)\, d\mu(x)$ indep of t.

$d\mu(x) =$ Lebesgue.

$\left. \begin{array}{l} y = x+t \\ dy = dx \end{array} \right\}$

Other group is $(\mathbb{R}_+^\times, \times)$. Again want

$\{x > 0\}$

invariant (Haar) measure:

i.e. if fix $t>0$, $\int f(x\cdot t)\, d\mu^{\times}(x)$ ← invariant under t, (times)

Set $d\mu^{\times}(x) = \boxed{\frac{dx}{x}} = \frac{dy}{y}$ $\quad \mathbb{R}^{\times}_{+}$ $\quad y = x\cdot t$, $dy = t\cdot dx$.

Haar measure on $(\mathbb{R}^{\times}_{+}, \times)$

Try to find an invariant measure on $(\mathbb{Q}_p, +)$.

$\mu(\mathbb{Z}_p) = 1$. ← uncountable. Try starting with a normalization.

$\mathbb{Z}_p = \bigsqcup_{a_0=0}^{p-1} \underbrace{(a_0 + p\cdot\mathbb{Z}_p)}$ ← all translations of one-another, no other choice

but $\mu(a_0 + p\mathbb{Z}_p) = \frac{1}{p}$.

Iterate down, $\mu(a_0 + a_1 p + a_2 p^2 + p^3\mathbb{Z}_p) = \frac{1}{p^3}$.

$p^{-1}\mathbb{Z}_p = \bigsqcup_{a_{-1}=0}^{p-1} (a_{-1}\cdot p^{-1} + \mathbb{Z}_p) \Rightarrow \mu(p^{-1}\mathbb{Z}_p) = p$.

$$\Rightarrow \mu(a + p^n \mathbb{Z}_p) = p^{-n}. \qquad \text{invariant under translation.}$$

$a \in \mathbb{Q}_p$, $n \in \mathbb{Z}$

__Exercise:__ is $\mathbb{Q}_p \setminus \mathbb{Q}$ measurable?.

__Ex:__ $\mu(\mathbb{Z}_p^\times) = ?. = \frac{p-1}{p}.$

$$\mathbb{Z}_p^\times = \{x \in \mathbb{Q}_p : |x|_p = 1\} = \{\underbrace{a_0 \cdot p^0 + a_1 p^1 + a_2 \cdot p^2 + \dots}_{?}\}, \quad a_0 \neq 0.$$

__Ex:__ $\mu(\mathbb{Z}_p \setminus \{0\}) = \sum_{n \geq 0} \mu(p^n \cdot \mathbb{Z}_p^\times) = \frac{p-1}{p} \sum_{n \geq 0} p^{-n}$

$$\mathbb{Z}_p = p \cdot \mathbb{Z}_p \sqcup \mathbb{Z}_p^\times = \mathbb{Z}_p^\times \sqcup (p \cdot \mathbb{Z}_p \sqcup p^2 \cdot \mathbb{Z}_p$$

$$\mathbb{Z}_p \setminus \{0\} = \bigsqcup_{n \geq 0} p^n \cdot \mathbb{Z}_p^\times$$

$$\underbrace{0 \cdot 1 + 0 \cdot p + 0 \cdot p^2 + \dots + \overset{p-1 \text{ choices}}{a_n} \cdot p^n + a_{n+1} p^{n+1}}$$

$\mu(p^n \mathbb{Z}_p^\times) = (p-1)\, p^{-n-1}$ $\quad p^n \mathbb{Z}_p^\times$

$= \frac{p-1}{p} \cdot \frac{1}{1-\frac{1}{p}} = 1.$

Fix any $a \in \mathbb{Q}_p$.

$a = a_{-N} p^{-N} + \cdots + a_{-1} p^{-1} + a_0 + \cdots$

$\{a\} \subset \underbrace{a_{-N} p^{-N} + \cdots + a_k p^k}_{\text{fixed}} + p^{k+1} \mathbb{Z}_p$ for every k.

$\mu(\{a\}) = 0$. $\quad \mu(\cdot) = p^{-k} \to 0$; $k \to \infty$.

To integrate, we need functions $f : \mathbb{Q}_p \to \mathbb{C}$.

(Total aside: for which x does e^x exist in $\mathbb{Q}_p$? "$1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$" $\in \mathbb{Q}_p$?),

$f(x) = |x|_p^s$, "$\underbrace{\text{Mellin transform}}_{\text{generic function}}$"; Fix $s \in \mathbb{C}$

$x \in \mathbb{Q}_p$

Analogy: $e^{2\pi i x}$ is generic on $\mathbb{R}$.

(Fourier analysis).

What can we say about $f(x) = |x|_p^s$?

This f is locally constant (except at 0),

i.e., $\forall x \in \mathbb{Q}_p$, $\exists\, U$ open $\ni x$ s.t. $f|_U \equiv f(x)$.

Compute:
$$\int_{\mathbb{Z}_p} |x|_p^s \, dx = \int_{\mathbb{Z}_p \setminus \{0\}} |x|_p^s \, dx$$

$$\mathbb{Z}_p \setminus \{0\} = \bigsqcup_{n \geq 0} p^n \mathbb{Z}_p^\times \qquad = \sum_{n \geq 0} \int_{p^n \mathbb{Z}_p^\times} \underbrace{|x|_p^s}_{|x|_p = p^{-n}} \, dx$$

$$= \sum_{n \geq 0} p^{-ns} \underbrace{\mu(p^n \mathbb{Z}_p^\times)}_{(p-1)\, p^{-n-1}}.$$

$$= \frac{p-1}{p} \sum_{n \geq 0} p^{-n(s+1)}$$

Abs conv $\mathrm{Re}\, s > -1$,

$$= \frac{p-1}{p} \cdot \frac{1}{1-p^{-(s+1)}}.$$

What about $(\mathbb{Q}_p^\times, \times)$ Haar measure here?

Want: Fix $t \in \mathbb{Q}_p^\times$,

$$\int_{\mathbb{Q}_p} f(x \cdot t)\, d^\times x = \int_{\mathbb{Q}_p} f(x)\, d^\times x$$

$$\boxed{d^\times x = \frac{dx}{|x|_p} \cdot c}$$

Given set $\overset{\mathbb{Q}_p}{a} + p^n \mathbb{Z}_p$, $t \in \mathbb{Q}_p^\times$,

$$\mu(t(a+p^n\mathbb{Z}_p)) = p^{-t} \mu(a+p^n\mathbb{Z}_p) \qquad p^t \mathbb{Z}_p^\times,$$

Want $1 = \mu^\times(\mathbb{Z}_p^\times) = \int_{\mathbb{Z}_p^\times} \frac{dx}{|x|_p} c = c \cdot \frac{(p-1)}{p}$, $\quad c = \frac{p}{p-1}$

Ex:

$$\int_{\mathbb{Z}_p\setminus\{0\}} |x|_p^s \, dx = \int_{\mathbb{Z}_p\setminus\{0\}} |x|_p^{s-1} \frac{dx}{\cancel{|x|_p}} \cdot \frac{p}{p-1}$$

$$= \frac{\cancel{p}}{p-1} \cdot \frac{\cancel{p-1}}{\cancel{p}} \cdot \boxed{\frac{1}{1-p^{-s}}}.$$

Back to $\mathbb{R}$: Fourier transform:

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{2\pi i x\cdot\xi} \, dx$$

$\leftarrow$ Schwartz.

Inversion:

$$\int \hat{f}(\xi) e^{-2\pi i x\cdot\xi}$$

$$f(x) = \int \hat{f}(\xi)\, e \; d\xi.$$

$\mathbb{R}$ $\qquad$ $\mathbb{Z}$

$e^{2\pi i x}$ <u>additive</u> <u>char</u> $(\mathbb{R},+) \to S^1 \subset \mathbb{C}$.

$$e^{2\pi i (x+y)} = e^{2\pi i x} \cdot e^{2\pi i y}, \quad e^{2\pi i n} = 1.$$

$\mathbb{Q}_p$ analogue?. $\quad \mathbb{Q}_p \to S^1 \subset \mathbb{C}$

$$f(x) = e^{2\pi i x} \quad \underline{13} \text{ loc const},$$

$$f(a_{-N}p^{-N} + \cdots + \overbrace{a_0 p^0 + a_1 p^1 + \cdots\cdots}).$$

$$= e^{2\pi i (a_{-N}p^{-N} + \cdots + a_{-1}p^{-1})} \in \overset{S^1}{\mathbb{C}}$$

- trivial on $\mathbb{Z}_p$ ✓.
- homom? $e^{2\pi i(x+y)} \overset{?}{=} e^{2\pi i x} \cdot e^{2\pi i y}$?

Yes!

Analogue of "Schwartz"

<u>loc-const</u>, <u>cpt support</u>.

$$\Rightarrow \text{finite sum of } \sum_{j=1}^{N} c_j \mathbb{1}_{\{a_j + p^{u_j}\mathbb{Z}_p\}}$$

Next time:

$$\widehat{\mathbb{1}_{\{a+p^n\mathbb{Z}_p\}}}(\xi) = \int_{\mathbb{Q}_p} \mathbb{1}_{\{a+p^n\mathbb{Z}_p\}} \cdot e^{2\pi i x\cdot\xi}\, dx.$$