Last time: $\mathfrak{g}$ Lie algebra of $G = SL_2(\mathbb{R})$

$\mathcal{U}(\mathfrak{g}) \leftarrow$ all order differential operators,

$Z(\mathcal{U}(\mathfrak{g})) =$ center, has invariant differential ops'

commutes with regular rep (both left & right).

f has K-type ℓ if: $f(g k_\theta) = e^{2\pi i \ell \theta} f(g)$,

$\forall g \in G$, $k_\theta \in K = SO(2)$.

generated by Casimir $\mathcal{C}$, $\leftarrow$ 2nd order invariant operator

$\mathcal{C}\Big|_{G/K} = \Delta = -y^2(\partial_{xx} + \partial_{yy})$, for $f(a_x a_y k_\theta)$ coordinates.

$R = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ (or conjugates thereof in $\mathfrak{g}_{\mathbb{C}}$).

& $L = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ act as Maass raising & lowering operators,

C in NAK coords is: $-y^2(\partial_{xx}+\partial_{yy}) + y\partial_y \cdot \partial_\theta$.

$L \overset{(?)}{=} -(z-\bar{z})\partial_{\bar{z}} + \partial_\theta$,

if $L|_{K\text{-type } \ell} = -(z-\bar{z})\partial_{\bar{z}} + \ell$.

R: K-type $\ell \to K$-type $\ell+2$

i.e. If $f(gk_\theta) = e^{i\theta\ell} f(g)$, then

$(Rf)(gk_\theta) = e^{i\theta(\ell+2)} f(g)$. Sim with L, $\ell \mapsto \ell-2$.

(Roughly)

<u>Big Idea</u>: (Gelfand-Graev-Piatetski-Shapiro)

Lift f Maass cusp form for $SL_2(\mathbb{Z})\backslash \underbrace{SL_2(\mathbb{R})/SO(2)}_{\mathbb{H}}$

to ϕ on $SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})$

Lift ϕ to

V_ϕ ∞ dim'l vector space irred under G-action

$f : \Gamma\backslash\mathbb{H} \to \mathbb{C}$ How to lift to G?

$\phi : \Gamma\backslash G \to \mathbb{C} : \gamma g \mapsto f(\underbrace{\gamma g \cdot i}_{\in\mathbb{H}}) = f(g\cdot i)$

$$\boxed{\begin{array}{l}\text{If } f \text{ is modular of wt } k. \quad f(z)\cdot y^{k/2} \\ \phi : \Gamma\backslash G \to \mathbb{C} : g \mapsto (\operatorname{Im} g\cdot i)^{k/2} f(g\cdot i)\end{array}}$$

$\phi \in L^2(\Gamma\backslash G)^k$

Let $V_\phi = \operatorname{span}_{g\in G} \underbrace{\pi(g).\phi}$

$\underbrace{\phi} \in \underbrace{L^2(\Gamma\backslash G)}$

$(\pi(g).\phi)(\gamma h) = \phi(\gamma h\cdot g)$,

(π, V_ϕ) is an irred G-rep.

$\mathcal{C}\phi \left(= \Delta f\right) = \lambda\phi \qquad \mathcal{C}(\pi(g).\phi) = \pi(g)\,\mathcal{C}\phi$

$= \lambda\cdot\pi(g).\phi.$

Conversely, if (π, V) $\subset L^2(\Gamma\backslash G)$ is an irred G-rep,

then $\mathcal{C}$ acts on V, Schur $\Rightarrow$ $\mathcal{C}$ acts by scalars.

If $\exists\, \phi \in V^K$ (then V is called "spherical").

$\pi(k).\phi = \phi \quad \forall k \in K.$ $\qquad C\phi = \lambda\phi \Rightarrow \phi$ is Maass form for Γ.

$\phi(g \cdot k_\theta) = \phi(g).$

Can take any $v \in V = \operatorname{span} \pi(g).\phi$ & take Fourier transform in K. i.e. $v(n_x a_y k_\theta)$

$= \sum a_\ell \cdot v_\ell(n_x a_y k_\theta)$, where $v_\ell(g\, k_\theta) = e^{2\pi i \ell \theta}\, v_\ell(g)$.

i.e. V is broken into K-isotypic components V_ℓ.

f on $\mathbb{H}$

$\downarrow$

ϕ on G $\in V$

$V = \bigoplus_{n \in \mathbb{Z}} R^n \phi$

$\vdots$		$\vdots$	$\lambda < \frac{1}{4}$ → or Complementary
$R^2\phi$	———	$\ell = 4$	
$R\phi$	———	$\ell = 2$	
ϕ	———	$\ell = 0$,	V^K
$L\phi$	———	$\ell = -2$	"principal" series reps ← $\lambda \geq \frac{1}{4}$
$L^2\phi$	———	$\ell = -4$	

If f was a modular form, $\phi = y^{k/2} f(z)\ (z \to g\cdot i)$.

$R\phi$ ——— $\ell = k$

ϕ ——— $\ell = k$ "lowest weight" vector

$\vdots$

V_ϕ 0 ——— $\ell = 2$

0 ——— $\ell = 0$

"discrete series rep"

$L\phi = 0$. $L = -(z-\bar{z})\partial_{\bar{z}} + K$.

$\Updownarrow$

$\partial_{\bar{z}} f = 0 \iff f$ holo.

'46 '47 Bargmann Gelfand-Naimark gave classification of unitary irred rep of $SL_2(\mathbb{R})$.

$\hookrightarrow$ gives nice models for each reps.

$$f(x+iy) = \sum a_n W_{\ell,n}(y)\, e(nx).$$

$\hookrightarrow \phi \rightsquigarrow V_\phi$ $\exists$ nice models in which to calculations

E.g. line model: $s \in \mathbb{C}$

$$V_s = \left\{ f : \mathbb{R} \to \mathbb{C} : \left(\pi\begin{pmatrix} a & b \\ c & d \end{pmatrix} f\right)(x) = (cx+d)^s f\left(\frac{ax+b}{cx+d}\right) \right\}$$

$\int |f|^2 < \infty$ $\quad \forall f \in V_s, \quad \Delta f = \underbrace{s(1-s)}_{\lambda} \cdot f.$

$B = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ "Borel parabolic" of $SL_2(\mathbb{R})$.

$= \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix}$. rep of B $\quad \chi\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = a^s$.

$$\mathrm{Ind}_B^G = \{ f(ga) = \chi(a) f(g) \}$$

$$G = \bar{N} \underbrace{AN}_{B}.$$

Back to adeles:

$$GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A})$$

$$L^2_\Delta(GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}))$$

Proved fund dom:

$$= L^2(GL_2(\mathbb{Z}) \backslash GL_2(\mathbb{R})) \times \prod_p GL_2(\mathbb{Z}_p)$$

Recall for $GL_1(\mathbb{Q}) \backslash GL_1(\mathbb{A})$, we had a method to lift χ Dirichlet char to ω Hecke char

Lift of Maass form $f(x+iy)$ for $SL_2(\mathbb{Z})$, to $GL_2(\mathbb{A})$ is <u>trivial</u>:

$$\phi\left(\overset{\psi}{g}\right) = f(x_\infty + i y_\infty)$$

$\forall g \in GL_2(\mathbb{A})$, $\exists \gamma \in GL_2(\mathbb{Q})$, $h_\infty \in GL_2(\mathbb{R})$,
$h_p \in GL_2(\mathbb{Z}_p)$

s.t. $$g = \gamma_\Delta \cdot (h_\infty, h_2, h_3, \dots).$$

$$h_\infty = \begin{pmatrix} y_\infty & x_\infty \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} r_\infty & 0 \\ 0 & r_\infty \end{pmatrix} k_\infty$$

When f on $\Gamma_0(N)$ with central char ψ,
need to be more careful with lift.
$(\mathfrak{g}, K)$-module acting at ∞-place,
$GL_2(\mathbb{A}_f)$ acting at finite places,

act say by $\left(I, \dots, \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, I, \dots\right).\phi = \lambda_p \cdot \phi$.

This is the Hecke operator, so

irreducible adelic representations capture not just eigenfunctions of Casimir (at ∞). also capture all Hecke operators in one fell swoop!

& define L-functions, study their FEs...

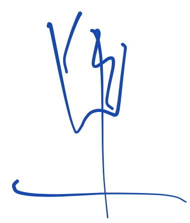
What about $GL_3(\mathbb{Z}) \backslash GL_3(\mathbb{R})$ Fund Domain

$g \in GL_3(\mathbb{R})$, $g = \underbrace{\begin{pmatrix} 1 & x_1 & x_3 \\ & 1 & x_2 \\ & & 1 \end{pmatrix}} \begin{pmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix} k$, $k \in O(3)$ / $SO(3)$

$f : \ _{f}\backslash G$

not abelian!

$K \in ?$



$u(g)$

$$\begin{pmatrix} 1 & x_1 & x_3 \\ & 1 & x_2 \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & u_1 & u_3 \\ & 1 & u_2 \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & \underbrace{u_1+x_1} & u_3+x_1u_2+x_3 \\ & 1 & \underbrace{u_2+x_2} \\ & & 1 \end{pmatrix}$$

$\zeta_1, \zeta_2,$

$$f = \underbrace{\sum_{\gamma \in G(\mathbb{Z})}} \sum_{n_1, n_2} a_{n_1,n_2} \cdot \underbrace{W_{n,\gamma}(y's)}_{\text{Jacquet Whittaker}} e(n_1x_1+n_2x_2)$$

$$L(f,s) = \sum \frac{\underbrace{a_{n_1,1}}}{n_1^s} = \prod_p \left(1-\frac{\alpha_p}{p^s}\right)^{-1}\left(1-\frac{\beta_p}{p^s}\right)^{-1}\left(1-\frac{\gamma_p}{p^s}\right)^{-1}$$

Godement-Jacquet std L-function.

$$L(\tilde{f},s) = \sum \frac{a_{1,n_2}}{n_2^s} \ldots$$

φ on GL_2 Maass α_p, β_p

$$\boxed{\sum \frac{a_{n_1,n_2}}{n_1^s n_2^w} = L(s,w).}$$

$$L(\mathrm{sym}^2\varphi, s) = \prod_p \left(1-\frac{\alpha_p^2}{p^s}\right)^{-1}\left(1-\frac{\alpha_p\beta_p}{p^s}\right)^{-1}\left(1-\frac{\beta_p^2}{p^s}\right)^{-1}$$

$$= \sum \frac{a_n}{n^s}.$$

Converse thm $\Rightarrow$ $\mathrm{sym}^2\varphi \rightsquigarrow f$ on GL_3.