Last time: φ Maass cusp form on $\Gamma_0(N)$,

$$\varphi(x+iy) = \sum_{n\geq 1} a_n W_{\lambda,n}(y)\, e(nx), \quad W_{\lambda,n}(y) = y^{1/2} K_{ir}(2\pi n y)$$

$$\boxed{\lambda = \tfrac14 + r^2}$$

$$W_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \notin \Gamma_0(N), \quad \psi(z) := \varphi(W_N z).$$

ψ on $\Gamma_0(N) \backslash \mathbb{H}$

$$L(\varphi, s) = \sum_{n\geq 1} \frac{a_n}{n^s}, \quad \text{FE:}$$

Integral representation:

$$\underbrace{N^{\frac{s}{2}} (2\pi)^{-s} \cdot L(\varphi,s)\, \Gamma\left(\frac{s+ir}{2}\right)\Gamma\left(\frac{s-ir}{2}\right)}_{} = \underbrace{N^{\frac{s}{2}} \int_0^\infty \varphi(iy)\, y^{-1/2} y^{s} \frac{dy}{y}}_{= \Lambda(\varphi, s).}$$

$$= N^{\frac{1-s}{2}} \Lambda(\psi, 1-s).$$

When level $N=1$, this L-function $\underline{\text{FE}}$ is enough to ensure that φ on $SL_2(\mathbb{Z}) \backslash \mathbb{H}$

What to do $N>1$?. Suffices to know about FE of twisted L-functions.

Let $(D,N)=1$, χ prim char mod D.

"Rankin-Selberg" $GL(2)\times GL(1)$

Want: $L(\varphi \otimes \chi, s) := \sum_{n\geq 1} \frac{\boxed{a_n \cdot \chi(n)}}{n^s}$

$\leftarrow \alpha_p^{(1)}, \alpha_p^{(2)}$

$$L(\varphi,s) = \prod_p \prod_{j=1}^{d} \left(1 - \frac{\alpha_p^{(j)}}{p^s}\right)^{-1}$$

Aside & $L(\psi,s) = \prod_p \prod_{\ell=1}^{m} \left(1 - \frac{\beta_p^{(\ell)}}{p^s}\right)^{-1}$

$$L(\varphi\times\psi, s) = \prod_p \prod_{j=1}^{d} \prod_{\ell=1}^{m} \left(1 - \frac{\alpha_p^{(j)} \beta_p^{(\ell)}}{p^s}\right)^{-1}$$

$$= \prod_p \underbrace{\left(1 - \frac{\alpha_p^{(1)}\chi(p)}{p^s}\right)^{-1}\left(1 - \frac{\alpha_p^{(2)}\chi(p)}{p^s}\right)^{-1}}_{\left(1 - \frac{a_p \cdot \chi(p)}{p^s} + \frac{\chi(p)^2}{p^{2s}}\right)^{-1}}.$$

Consider: $\varphi_\chi(z) := \sum_{n\geq 1} a_n \underline{\chi(n)}\, W_{\lambda,n}(y)\, e(nx)$.

Then: $\int_0^\infty \varphi_\chi(iy)\, y^{-1/2} y^s \frac{dy}{y} = \Lambda(\varphi\times\chi, s)$.

$\nwarrow$ FE?.

Does φ_χ have any invariance??.

Assumed: φ itself is invariant under $\Gamma_0(N)$.

trivial nebentypus

$$\hat{\chi}(m) = \frac{1}{\sqrt{D}} \sum_{n \pmod D}{}' \chi(n) e^{\frac{2\pi i n m}{D}} = \overline{\chi(m)} \frac{\tau(\chi)}{\sqrt{D}}$$

$$\boxed{\tau(\chi) = \sum_{n(D)} \chi(n) e^{\frac{2\pi i n}{D}}}$$

$$\chi(n) = \frac{1}{\sqrt{D}} \sum_{m \pmod D}{}' \hat{\chi}(m) e^{-\frac{2\pi i n m}{D}} = \boxed{\chi(-1)\frac{\tau(\chi)}{D} \sum_{m(D)}{}' \overline{\chi(m)} e^{\frac{2\pi i n m}{D}}}$$

$$\underbrace{\varphi_\chi(z) = \chi(-1)\frac{\tau(\chi)}{D} \sum_{m(D)}{}' \overline{\chi(m)}} \underbrace{\underbrace{\sum_{n \geq 1} a_n W_{1,n}(y)\, e\left(n\left(x + \frac{m}{D}\right)\right)}}.$$

$$\varphi\left(\underbrace{\begin{pmatrix} D & m \\ 0 & D \end{pmatrix} \cdot z}\right)$$

For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(ND^2) \cap \Gamma_1(D)$.

$$\frac{Dz + m}{D} = z + \frac{m}{D}.$$

$$\varphi_\chi(\gamma z) = \frac{\chi(-1)\tau(\chi)}{D} \sum_{m(D)}{}' \overline{\chi(m)}\, \varphi\left(\underbrace{\begin{pmatrix} D & m \\ 0 & D \end{pmatrix} \gamma \overbrace{\begin{pmatrix} \frac{1}{D} & \frac{-m}{D^2} \\ 0 & \frac{1}{D} \end{pmatrix}}^{I} \begin{pmatrix} D & m \\ 0 & D \end{pmatrix} z}\right),$$

If $\gamma \in \Gamma_0(N)$, $c \equiv 0 (N)$.

Assume $c \equiv 0 (D^2)$.

Assume $a \equiv d \equiv 1 (D)$.

$$\underbrace{\begin{pmatrix} a + \frac{cm}{D} & b + \frac{m}{D}(d - a) - \frac{cm^2}{D^2} \\ c & d - \frac{cm}{D} \end{pmatrix}}_{\in\, \Gamma_0(N)}.$$

[Assume φ on $\Gamma_0(N)$]

Lemma: If $\gamma \in \Gamma_0(ND^2) \cap \Gamma_1(D) \supset \Gamma(ND^2)$.

Then $\varphi_\chi(\gamma z) = \varphi_\chi(z)$!

For FE of $L(\varphi_\chi, s) = L(\varphi \otimes \chi, s)$,

used $\psi(z) := \varphi(W_N z)$, i.e. $\varphi(z) = \psi(W_N^{-1} z)$.

$$W_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}, \qquad W_N^{-1} = \begin{pmatrix} 0 & \frac{1}{N} \\ -1 & 0 \end{pmatrix}$$

Consider: $\varphi_\chi\left(\begin{pmatrix} 0 & -1 \\ ND^2 & 0 \end{pmatrix} z\right)$

$$= \frac{\chi(-1)\,\tau(\chi)}{D} \sum'_{m\,(D)} \overline{\chi(m)}\, \varphi\left(\begin{pmatrix} D & m \\ 0 & D \end{pmatrix}\begin{pmatrix} 0 & -1 \\ ND^2 & 0 \end{pmatrix} z\right).$$

$$= \frac{\chi(-1)\,\tau(\chi)}{D} \sum'_{m\,(D)} \overline{\chi(m)}\, \psi\left(\underbrace{W_N^{-1}\begin{pmatrix} D & m \\ 0 & D \end{pmatrix}\begin{pmatrix} 0 & -1 \\ ND^2 & 0 \end{pmatrix}\overbrace{\begin{pmatrix} \frac{1}{D} & \frac{-\ell}{D^2} \\ 0 & \frac{1}{D} \end{pmatrix}\begin{pmatrix} D & \ell \\ 0 & D \end{pmatrix}}^{I\ \forall \ell}}_{} z\right)$$

$$\begin{pmatrix} D & -\ell \\ -mN & \frac{1+\ell m N}{D} \end{pmatrix} \sim \begin{pmatrix} D^2 & -D\ell \\ -DmN & 1+\ell m N \end{pmatrix}.$$

If $1+\ell mN \equiv 0\ (D)$, i.e. $\ell \equiv -\overline{m}\,\overline{N}\ (D)$.

then $\begin{pmatrix} D & -\ell \\ -m\underline{N} & \frac{1+\ell m N}{D} \end{pmatrix} \in \Gamma_0(N)$.

$$= \frac{\chi(-1)\tau(\chi)}{D} \underset{m(D)}{\sum\nolimits'} \overline{\chi(m)}\, \psi\left(\begin{pmatrix} D & \ell \\ 0 & D \end{pmatrix} z\right).$$

Change $\ell = -\overline{m}\,\overline{N}\ (D)$, $m = -\overline{\ell}\,\overline{N}$

$$= \frac{\cancel{\chi(-1)}\tau(\chi)}{D} \underset{\ell(D)}{\sum\nolimits'} \overline{\chi(\cancel{-}\overline{\ell}\,\overline{N})}\, \psi\left(\begin{pmatrix} D & \ell \\ 0 & D \end{pmatrix} z\right)$$

$\psi_\chi(W_{ND^2} z)$.

$$= \frac{\chi(N)\tau(\chi)}{D} \underset{\ell(D)}{\sum\nolimits'} \chi(\ell)\, \psi\left(\begin{pmatrix} D & \ell \\ 0 & D \end{pmatrix} z\right) \,(=$$

Recall:

$$\psi_\chi(z) = \frac{\chi(-1)\tau(\chi)}{D} \underset{m(D)}{\sum\nolimits'} \overline{\chi(m)}\, \psi\left(\begin{pmatrix} D & m \\ 0 & D \end{pmatrix} z\right)$$

$$\parallel$$

$$\sum_{n\geq 1} a_n \chi(n)\, W_{f,n}(y)\, e(nx).$$

$$\boxed{\psi_\chi(z) := \sum_{n\geq 1} b_n \chi(n) W_{\lambda,n}(y)\, e(nx)}$$

$$\boxed{\psi_\chi(w_{D^2N} z) = \chi(-1) \frac{\tau(\chi)}{\tau(\bar\chi)} \chi(N)\, \psi_{\bar\chi}(z)}$$

$$\int_0^\infty \varphi_\chi(iy)\, y^{s-1/2} \frac{dy}{y} = \Lambda(\varphi\otimes\chi, s).$$

$$y \mapsto \frac{1}{ND^2 y}$$

$$(ND^2)^{\frac12 - s} \int_0^\infty \underbrace{\varphi_\chi\left(\frac{i}{ND^2 y}\right)} y^{\frac12 - s} \frac{dy}{y} \qquad \boxed{\tau(\chi)\tau(\bar\chi) = D,}$$

$$= (ND^2)^{\frac12 - s} \chi(-1) \frac{\tau(\chi)}{\tau(\bar\chi)} \chi(N) \underbrace{\int_0^\infty \psi_{\bar\chi}(iy)\, y^{\frac12 - s} \frac{dy}{y}},$$

If φ on $\Gamma_0(N)$ & $(D,N)=1$, $\chi(D)$ primitive. $\Lambda(\varphi\otimes\bar\chi, 1-s)$.

Thm: $(ND^2)^{\frac{s}{2}} \Lambda(\varphi\otimes\chi, s) = (ND^2)^{\frac{1-s}{2}} \chi(-N) \frac{\tau(\chi)^2}{D} \Lambda(\varphi\otimes\bar\chi, 1-s)$

Weil's Converse Thm (1967?): Let $a, b : \mathbb{N} \to \mathbb{C}$, $a_n, b_n \ll n^c$.

Assume $\exists\, N \geq 1$ $\left(\& \ \lambda = \frac{1}{4} + r^2\right)$ s.t. $\forall\, (D, N) = 1$, $\forall\, \chi(D)$ prim,

$$L_1(s, \chi) := \sum_{n \geq 1} \frac{a_n \chi(n)}{n^s}$$

$$L_2(s, \chi) := \sum_{n \geq 1} \frac{b_n \chi(n)}{n^s}$$

& have entire analytic cont, bounded in vertical strips,

and FE:

$$(ND^2)^{\frac{s}{2}} \Lambda_1(\chi, s) = (ND^2)^{\frac{1-s}{2}} \chi(-N) \frac{\tau(\chi)^2}{D} \Lambda_2(\bar{\chi}, 1-s).$$

<u>Then</u> $\varphi(z) := \sum_{n \geq 1} a_n W_{\lambda, n}(y)\, e(nx)$ is $\Gamma_0(N)$-invariant i.e. a Maass form.