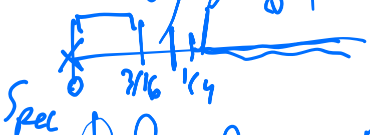
Last time: Selberg 1/4 Conj:

If φ is a Maass form on congruence subgp of $SL_2(\mathbb{Z})$, then $\lambda_\varphi \geq \frac{1}{4}$ (where $\Delta\varphi = \lambda_\varphi \cdot \varphi$)

$\not\equiv c.$



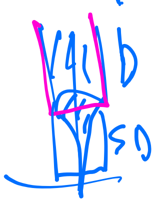
Pf for $\Gamma = SL_2(\mathbb{Z})$ i.e. level 1: Let φ be a Maass cusp form,

$(\lambda = \frac{1}{4} + r^2)$.

$$\varphi(x+iy) = \sum_{n\geq 1} \underbrace{a_n(y)}_{a_n \cdot y^{1/2} K_{ir}(2\pi n y)} \cdot e(nx).$$

$\Delta = -y^2(\partial_{xx} + \partial_{yy})$

$$\lambda \|\varphi\|^2 = \langle \Delta\varphi, \varphi\rangle = + \int_D |\nabla\varphi|^2 \, dx\,dy = + \int_D |(\partial_x + \partial_y)\varphi|^2 \, dx\,dy$$



$$= \frac{1}{2} \int_{D \cup SD} |\nabla\varphi|^2 \, dx\,dy$$

$$\geq \frac{1}{2} \int_{\frac{\sqrt{3}}{2}}^{\infty} \underbrace{\int_{-1/2}^{1/2} |\nabla\varphi|^2 \, dx}\, dy.$$

$$\nabla\varphi = \sum_{n\geq 1} \left(a_n(y) \cdot 2\pi i n + a_n'(y)\right) e(nx).$$

$$\underbrace{\int_0^1 |\nabla\varphi|^2 \, dx} = \sum_{n\geq 1} \left|a_n(y) \cdot 2\pi i n + a_n'(y)\right|^2 \quad \text{(Parseval)}$$

$$\geq \sum_{n\geq 1} |a_n(y)|^2 4\pi^2 n^2$$

$$\geq \sum_{n \geq 1} |a_n(y)|^2 (4\pi^2) \cdot \quad = \underbrace{4\pi^2 \int_{-1/2}^{1/2} |\varphi|^2 \, dx,}$$

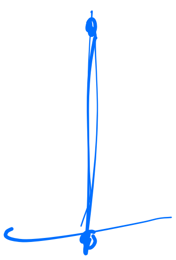
$$y > \frac{\sqrt{3}}{2}$$

$$\lambda \|\varphi\|^2 \geq \frac{4\pi^2}{2} \int_{\sqrt{3}/2}^{\infty} \int_{-1/2}^{1/2} |\varphi|^2 \, dx \, dy \left(\frac{\sqrt{3}}{2y}\right)^2$$

$$= \frac{3}{2}\pi^2 \cdot \underbrace{\int_{\sqrt{3}/2}^{\infty} \int_{-1/2}^{1/2}} |\varphi|^2 \frac{dx\,dy}{y^2}$$

$$\geq \frac{3}{2}\pi^2 \|\varphi\|^2. \Rightarrow \underline{\lambda \geq \frac{3}{2}\pi^2} \approx 14.8\ldots \quad \underline{\geq \frac{1}{4}.}$$

(truth: $\lambda_1 \approx 91\ldots$). $= \frac{1}{4} + r^2$, $r \approx 9.\ldots$



$$\int_0^\infty \varphi(iy)\, y^{-1/2} y^s \frac{dy}{y} = (2\pi)^{-s} \sum_{n \geq 1} \frac{a_n}{n^s} \underbrace{\int_0^\infty \cancel{y^{1/2}} K_{ir}(\cancel{2\pi n y})\, y^s \frac{dy}{y}}$$

$\varphi(iy)$

$= \varphi\left(\frac{i}{y}\right)$

$$\Gamma\left(\frac{s+ir}{2}\right)\Gamma\left(\frac{s-ir}{2}\right).$$

$$= (2\pi)^{-s}\,\Gamma \cdot \Gamma \cdot \underbrace{L(\varphi, s)}_{\prod_p \left(1 - \frac{\alpha_p}{p^s}\right)^{-1}\left(1 - \frac{\beta_p}{p^s}\right)^{-1},} = \Lambda(\varphi, s) = \Lambda(\varphi, 1-s).$$

If φ also Hecke-Maass form!

Conversely, what if we have a sequence $a : \mathbb{N} \to \mathbb{C}$, s.t. $a_n \ll n^c$. ($\Rightarrow$ $L(s) = \sum \frac{a_n}{n^s}$ converges $\mathrm{Re}\, s \gg 1$)

& suppose $L(s)$ has analytic cont to $\mathbb{C}$,

bounded in vertical strips, $\sigma_1 \le \mathrm{Re}\, s \le \sigma_2$,

& $\exists r$ FE $\Lambda(s) = (2\pi)^{-s}\,\Gamma\left(\frac{s+ir}{2}\right)\Gamma\left(\frac{s-ir}{2}\right) L(s) = \Lambda(1-s)$.

Converse

Thm: Then $\varphi(x+iy) := \sum_{n\geq 1} a_n\, y^{1/2} K_{ir}(2\pi n y)\, e(nx)$

is a Maass form for $SL_2(\mathbb{Z})$.

pf: Need $\Delta\varphi = \lambda\varphi$. ✓. $\varphi(x+iy) = \varphi(x+1+iy)$. ✓

Need: $\varphi\left(\frac{-1}{x+iy}\right) = \varphi(x+iy)$. ??? $\Gamma = \left\langle \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\rangle$.

$$\varphi(iy) = \frac{1}{2\pi i}\int_{(2)} \Lambda(s)\, y^{\frac{1}{2}-s}\, ds = \frac{1}{2\pi i}\int_{(c)} \Lambda(s)\, y^{\frac{1}{2}-s}\, ds$$

c big enough that $L = \Sigma$.

$$= \sum_{n\geq 1} a_n \frac{1}{2\pi i}\int_{(c)} y^{1/2} (2\pi)^{-s}\, \Gamma\left(\frac{s+ir}{2}\right)\Gamma\left(\frac{s-ir}{2}\right) n^{-s} y^{-s}\, ds.$$

$$= \sum_{n\ge 1} a_n\, y^{1/2} K_{ir}(2\pi n y)$$

$$\boxed{K_w(y) = \int_0^\infty e^{-y(u+1/u)}\, u^w \frac{du}{u}}$$

↳ OTOH $= \frac{1}{2\pi i}\int_{(2)} \Lambda(1-s)\, y^{\frac12 - s}\, ds$

$$\boxed{= \frac{1}{2\pi i}\int_{(2)} \Gamma\left(\frac{s+w}{2}\right)\Gamma\left(\frac{s-w}{2}\right) y^{-s}\, ds}$$

$$= \frac{1}{2\pi i}\int_{(c)} \Lambda(s)\left(\frac{1}{y}\right)^{\frac12 - s} ds. = \underbrace{\varphi\left(\frac{i}{y}\right) . = \varphi(iy)}$$

Let $\psi(z) := \varphi(z) - \varphi\left(\frac{-1}{z}\right).$

__Claim:__ $\Delta\psi = \lambda\psi$: $\Delta\psi = \Delta\varphi - \Delta\varphi\circ S$

$= \lambda\varphi - \lambda\varphi\circ S = \lambda\psi.$ ✓

& $\psi\big|_{\{iy\}} \equiv 0. \Rightarrow \psi \equiv 0.$ $\varphi(z) = \varphi\left(\frac{-1}{z}\right)$ ✓

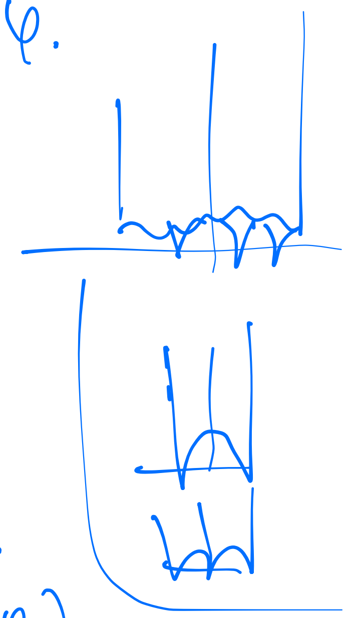
Consider φ a Maass cusp form on $\Gamma_0(N) < \Gamma(1).$

What can we do? $\begin{pmatrix}1&1\\0&1\end{pmatrix} \in \left\{\begin{pmatrix}*&*\\0&*\end{pmatrix} \bmod N\right\}$

$$\varphi(z) = \sum_{n\ge 1} a_n\, y^{1/2} K_{ir}(2\pi n y)\, e(nx).$$

✓ invariant under $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. ✓ $\Delta \varphi = \lambda \varphi$.

Other generators of $\Gamma_0(N)$?.



Same as before ⇒

$$\Lambda(\varphi, s) = \int_0^\infty \varphi(iy)\, y^{s-1/2} \frac{dy}{y} = (2\pi)^{-s}\Gamma\Gamma\, L(\varphi, s).$$

is entire

analytic cont is free. FE?.

Instead of $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, use $W_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$ ← has det N, not 1.

$W_N(z) = \dfrac{-1}{Nz}$ But $W_N \notin SL_2(\mathbb{Z})$, $W_N \notin \Gamma_0(N)$.

But Claim W_N normalizes $\Gamma_0(N)$: pf: let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, $c \equiv 0\ (N)$.

$$\underbrace{\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}}\begin{pmatrix} 0 & \frac{1}{N} \\ -1 & 0 \end{pmatrix} = W_N \begin{pmatrix} a & b \\ c & d \end{pmatrix} W_N^{-1}.$$

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{\det}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

$$\begin{pmatrix} -c & -d \\ Na & Nb \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{N} \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} d & -\frac{c}{N} \in \mathbb{Z} \\ \underbrace{-Nb}_{\equiv 0\,(N)} & a \end{pmatrix} \in \Gamma_0(N).$$

Let $\psi(z) := \varphi\left(\frac{-1}{\omega_N \cdot z}\right)$. $\frac{-1}{Nz} = \frac{-\bar{z}}{N|z|^2}$. Claim:

ψ is $d/_D$ Maass cuspform for $\Gamma_0(N)$.

$\Delta\psi = \lambda\psi$. ✓, $\psi(\gamma z) = \varphi(\omega_N \gamma z)$, $\gamma \in \Gamma_0(N)$

$\omega_N \gamma \cancel{\omega_N^{-1}} = \gamma_1 \omega_N$ $\qquad = \varphi(\cancel{\gamma_1} \omega_N z) = \psi(z)$.

$$= \sum_{n\geq 1} a_n \left(\frac{y}{N(x^2+y^2)}\right)^{1/2} K_{i\nu}\left(2\pi n \frac{y}{N(x^2+y^2)}\right) e(n\cancel{x}).$$

Instead: $\psi(z) = \sum_{n\geq 1} b_n y^{1/2} K(2\pi n y)\, e(nx)$.

$$\int_0^\infty \varphi(iy)\, y^{s-1/2} \frac{dy}{y} = \Lambda(\varphi, s).$$

$y = \frac{1}{Nu}$, $\quad \frac{dy}{y} = \frac{-1}{Nu^2} du \cancel{Nu}$

$$N^{\frac{1}{2}-s}\int_0^\infty \underbrace{\varphi\left(\frac{i}{Nu}\right)}_{\psi(iu)} (Nu)^{\frac{1}{2}-s}\,\frac{du}{u}.$$

$$\boxed{N^{\frac{1-s}{2}}\Lambda(\psi,1-s). = N^{\frac{s}{2}}\Lambda(\varphi,s).}$$

Can we reverse this argument like before?

i.e. assume $a,b : \mathbb{N}\to\mathbb{C}$, $a_n, b_n \ll n^c$,

$L_1(s) = \sum \frac{a_n}{n^s}$, $L_2(s) = \sum \frac{b_n}{n^s}$, analytic cont, FE.

Can I conclude that $\varphi(z) := \sum a_n \underline{y^{k/2} K(\;)}\, \underline{e(nx)}$

is invariant under $\Gamma_0(N)$?. (& $\psi = \sum b_n \ldots$),

How to capture generators of $\Gamma_0(N)$??

Need: Know not just one FE, need

"Twisted" FE's: Need to know that primitive Dir char χ conductor D, $(D,N)=1$,

Then $L(s,\chi) = \sum \frac{a_n \chi(n)}{n^s}$ also needs to have FE,

Next time: How to prove FE for ?