Last time: $f(z) = \sum_{n\geq 1} a_n\, y^{1/2} K_{ir}(2\pi n y)\, e(nx)$,

$f(\gamma z) = f(z)$ $\forall \gamma \in SL_2(\mathbb{Z})$

f is a Maass even cusp form, i.e. $\Delta = -y^2(\partial_{xx} + \partial_{yy})$, $\Delta f = \lambda f$,

$\lambda = \frac{1}{4} + r^2 = (\frac{1}{2} + ir)(\frac{1}{2} - ir)$, where $\boxed{f \in L^2(\Gamma \backslash \mathbb{H}). \quad L^2(SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R}) / SO(2))}$

$K_w(y) = \frac{1}{2}\int_0^\infty e^{-y(u + 1/u)}\, u^w \frac{du}{u}$.

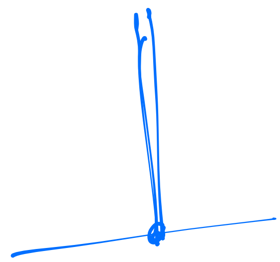
$w \in \mathbb{C}$? $y \in (0,\infty)$?

$f(x+iy) = \sum_{n\in\mathbb{Z}} a_n(y)\, e(nx)$, $\boxed{-y^2(-4\pi^2 n^2 a_n(y) + a_n''(y)) = \lambda \cdot a_n(y)}$

Exercise: verify that satisfies.

(1) Where does this integral rep come from?

(2) Want: $L(f,s) = \sum_{n\geq 1} \frac{a_n}{n^s}$?.



Consider: $\int_0^\infty f(iy)\, y^{-1/2}\, y^s \frac{dy}{y} = (2\pi)^{-s} \sum_{n\geq 1} \frac{a_n}{n^s} \int_0^\infty$

$\cancel{y^{-1/2}}\, \cancel{y^{1/2}}\, K_{ir}(\cancel{2\pi n}\, y)\, y^s \frac{dy}{y}$

$y \mapsto \frac{y}{2\pi n}$.

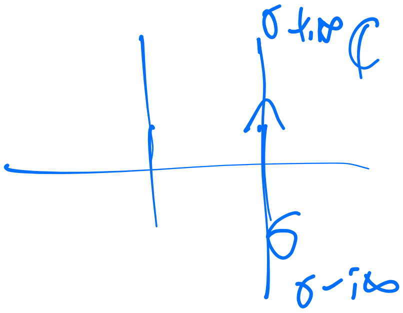
$= (2\pi)^{-s} \cdot L(f,s) \int_0^\infty K_{ir}(y)\, y^s \frac{dy}{y} \leftarrow ??$.

"soft" explanation of the Mellin transform.

$$\varphi : \underset{\mathbb{R}_+^\times}{\underset{\shortparallel}{(0,\infty)}} \to \mathbb{C}, \qquad \tilde{\varphi}(s) = \underbrace{\int_0^\infty \varphi(y) \cdot y^s \cdot \frac{dy}{y}}.$$

<u>Claim (Mellin inversion):</u> $I(y) := \dfrac{1}{2\pi i}\displaystyle\int_{(\sigma)} \tilde{\varphi}(s)\, y^{-s}\, ds = \varphi(y)$.

$(\sigma) \leftarrow \mathrm{Re}\, s = \sigma = 10$.



<u>pf 0:</u> Mellin $\longleftrightarrow$ log-Fourier $\Rightarrow$

<u>pf attempt 1:</u>

$$\frac{1}{2\pi i}\int_{(\sigma)} \left[\int_0^\infty \varphi(u)\, u^s \frac{du}{u}\right] y^{-s}\, ds$$

$$\cancel{=} \int_0^\infty \underset{\downarrow\ \varphi(y)}{\varphi(u)} \Bigg[\underset{\delta\delta?\ \ \delta_{u=y}}{\frac{1}{2\pi i}\int_{(\sigma)}} \underbrace{\left(\frac{u}{y}\right)^s}_{|\cdot| = \left|\frac{u}{y}\right|^{\mathrm{Re}\, s}} ds\Bigg] \frac{du}{u}$$

<u>pf attemp 2:</u> Integrate by parts: $\tilde{\varphi}(s) = -\displaystyle\int_0^\infty \varphi'(y)\, \frac{y^s}{s}\, dy$.

Then: $I(y) = \frac{1}{2\pi i}\int_{(\sigma)} \left[-\int_0^\infty \varphi'(u)\frac{u^s}{s}\,du\right] y^{-s}\,ds$

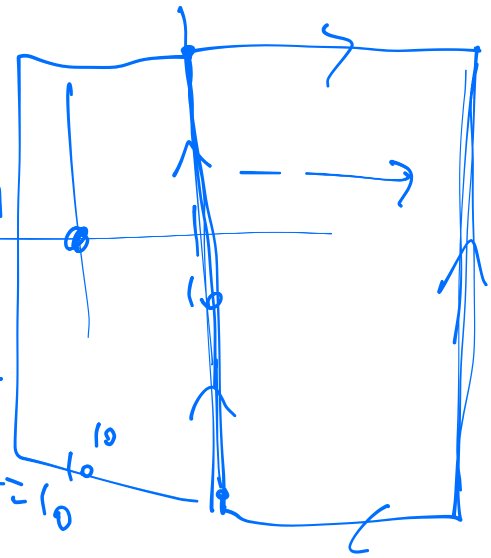
$$\cancel{=} -\int_0^\infty \varphi'(u)\left[\frac{1}{2\pi i}\int_{(\sigma)}\left(\frac{u}{y}\right)^s \frac{ds}{s}\right]du.$$

Perron integral: $\frac{1}{2\pi i}\int_{(10)} \boxed{\frac{t^s}{s}}\, ds = \begin{cases}0, & t<1\\ 1, & t>1\end{cases}$

$t>0$

$|\cdot| = t^{\mathrm{Re}\, s}$

If $t<1$, pull contour from $\sigma=10$ to $\sigma=10^{10}$



$$= -\int_y^\infty \varphi'(u)\,du. = \varphi(y).$$

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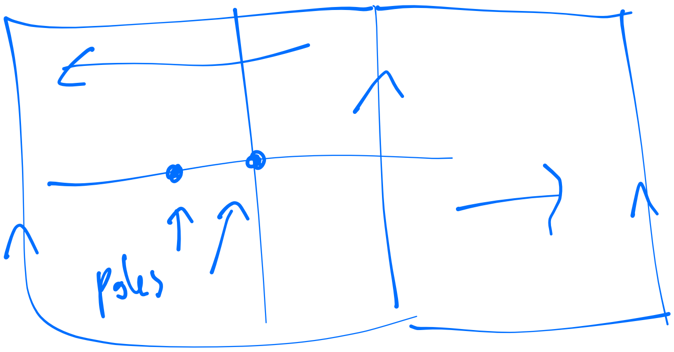
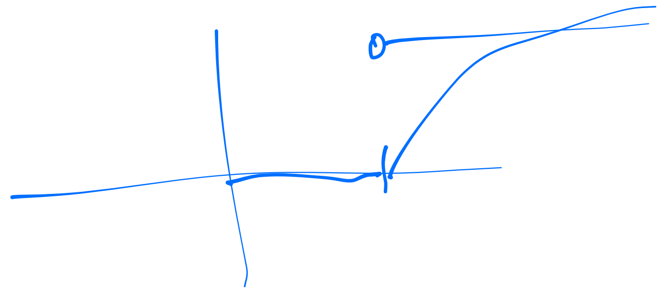
pf 3 (Goldfeld-K): $\tilde{\varphi}(s) = +\int_0^\infty \varphi''(y)\frac{y^{s+1}}{s(s+1)}\,dy$

$$I(y) = \frac{1}{2\pi i}\int_{(\sigma)}\left[\int_0^\infty \varphi''(u)\frac{u^{s+1}}{s(s+1)}\,du\right]y^{-s}\,ds$$

$$= \int_0^\infty \varphi''(u)\left[\frac{1}{2\pi i}\int_{(\sigma)}\left(\frac{u}{y}\right)^s\frac{ds}{s(s+1)}\right]u\cdot du.$$

Perron identity: $\displaystyle \frac{1}{2\pi i}\int_{(\sigma)} t^s \frac{ds}{s(s+1)} = \begin{cases} 0 , & t<1 \\ 1-\frac{1}{t} , & t>1 \end{cases}$





$$\to = \int_y^\infty \varphi''(u) \underbrace{\left(1-\frac{y}{u}\right) u}_{(u-y).} \, du = -\int_y^\infty \varphi'(u)\cdot 1 \cdot du = \varphi(y)$$

Mellin Convolution: $f, g : \mathbb{R}_+^\times \to \mathbb{C}$,

$$(f*g)(y) = \int_0^\infty f(u)\, g\left(\frac{y}{u}\right) \frac{du}{u}.$$

Exercise: $\widetilde{f*g}(s) = \tilde{f}(s)\cdot\tilde{g}(s).$

$u=y^2$

E.g.: $f_w(y) = e^{-y^2}\cdot y^w$, $\displaystyle \tilde{f}_w(s) = \int_0^\infty e^{-y^2} y^w y^s \frac{dy}{y} = \frac{1}{2}\Gamma\left(\frac{s+w}{2}\right).$

$$(f_w * f_{-w})(y) = \int_0^\infty e^{-u^2} \underbrace{u^w \, e^{-(y/u)^2} \left(\frac{y}{u}\right)^{-w}}_{\left(\frac{u^2}{y}\right)^w} \frac{du}{u}$$

Let $v = \frac{u^2}{y}$, $\frac{dv}{v} = \frac{2\cancel{u}\, du \cdot \cancel{y}}{\cancel{y}\, u^{\cancel{2}}}$

$$= \frac{1}{2} \int_0^\infty e^{-v \cdot y - y/v} \, v^w \frac{dv}{v} = \frac{1}{2} \int_0^\infty e^{-y(v + \frac{1}{v})} v^w \frac{dv}{v}.$$

$$= K_w(y) = K_{-w}(y). \text{ entire in } w.$$

Back to:

$$\int_0^\infty f(iy) y^{-1/2} \cdot y^s \frac{dy}{y} \overset{\text{Re } s \gg 1.}{=} (2\pi)^{-s} \cdot \underbrace{L(f, s)} \underbrace{\int_0^\infty K_{ir}(y) y^s \frac{dy}{y}}_{\frac{1}{4} \Gamma\left(\frac{s+ir}{2}\right) \Gamma\left(\frac{s-ir}{2}\right)}$$

$$\Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right) \overset{?}{=} \Gamma(s).$$

$f(\gamma z) = f(z)$ gives L-function FE. & analytic cont.

Is $L(f, s)$ really an L-function? Not until Euler prod.

In general: not necessarily need Hecke op's.

Look at $M \in \{ M_{2\times 2}(\mathbb{Z}),\ \det = N \} = \Delta_N$.

$\gamma \in \Gamma = SL_2(\mathbb{Z})$ acts on Δ_N, $\det(\gamma \cdot M) = 1 \cdot N$.

Q: "Fund dom" $\Gamma \backslash \Delta_N \overset{?}{=}$ coset reps?.

Idea: given $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Delta_N$, make it as "simple" as possible by left mult by $SL_2(\mathbb{Z})$.

$$\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{\in SL_2(\mathbb{Z})}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} & \\ \underbrace{cA+dC}_{=0?} & \end{pmatrix}$$

$(A, C) \neq (0,0)$ $(\Leftarrow N \neq 0)$.

$\Rightarrow \exists (c,d)=1$ s.t. $cA + dC = 0$,

$$\begin{pmatrix} * & * \\ c & d \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}, \quad \alpha \cdot \delta = N.$$

is this unique? No, can left-mult by

$$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} * & * \\ c & d \end{pmatrix} = \begin{pmatrix} * & * \\ c & d \end{pmatrix}, \quad \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta + n\delta \\ 0 & \delta \end{pmatrix}$$

Thm: $\Gamma\backslash\Delta_N = \left\{ \begin{pmatrix}\alpha & \beta \\ 0 & \delta\end{pmatrix} : \alpha\cdot\delta = N,\ \beta\in\{0,\dots,\delta-1\} \right\}$

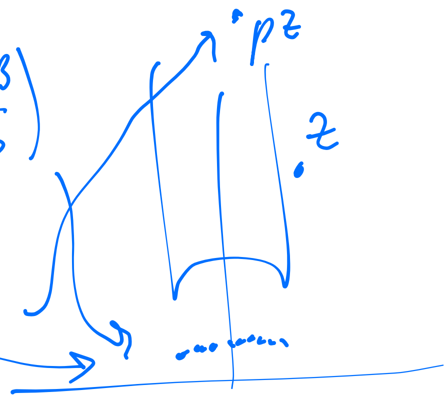
For $f\in L^2(\Gamma\backslash H)$:

Def: $$(T_N f)(\underset{\uparrow\,\gamma}{z}) = \frac{1}{N^{1/2}}\sum_{M\in\Gamma\backslash\Delta_N} f(M\underset{\uparrow\,\gamma}{z}) = \frac{1}{N^{1/2}}\sum_{\alpha\cdot\delta=N}\sum_{\beta=0}^{\delta-1} f\left(\frac{\alpha z+\beta}{\delta}\right)$$

Exercise: $T_N f \in L^2(\Gamma\backslash H)$.

Eg: $N=p$,
- $\alpha=1,\ \delta=p \;\to\; \sum_{\beta=0}^{p-1} f\left(\frac{z+\beta}{p}\right)$
- $\alpha=p,\ \delta=1 \;\to\; f(pz)$

"lowlying" horocycle for large p.



Exercise: $$T_N\circ T_M = \sum_{d\mid(N,M)} T_{\frac{N\cdot M}{d^2}}$$

$\Rightarrow$ T_N's commute!, Also commute with Δ.

So can take φ simultaneously eigenfunctions of Δ & all T_n's. "Hecke-Maass cuspforms".

$\Rightarrow$ for each φ, $L(\varphi, s)$ will have Euler products.

$$L^2_\rho(\mathbb{H}) \cap \{\Delta f = \underline{\lambda} f\}, \leftarrow \text{fin dim?}$$

fixed

Yes fin dim'l.

$\#\{\lambda < T\} = \ldots$