Last time: f, g modular forms of wt k,

$\langle f, g\rangle = \int_{D = \Gamma\backslash\mathbb{H}} f(z)\bar{g}(z) y^k \frac{dx\,dy}{y^2}$, $\Gamma = SL_2(\mathbb{Z})$

$M_k(\Gamma)$ = vector space of wt k mod forms.

f wt k, g wt l, $\int_{\Gamma\backslash\mathbb{H}} f\cdot\bar{g}\, y^{\frac{k+l}{2}} \frac{dx\,dy}{y^2}$.

$M(\Gamma) = \oplus_k M_k$ $\ni f, g$.

Algebraic $\leftrightarrow$ Rep Th.

φ Maass form on GL_2, $\varphi(x+iy) = \sum a_n W_n(y) e(nx)$.

$L(\varphi, s) = \sum_{n\geq 1} \frac{a_n}{n^s} = L(\pi, s)$ $(\pi, V) \circlearrowleft L^2(\Gamma\backslash G)$, irrep of $SL_2(\mathbb{R})$

$|a_p| \ll p^{1/2}$
$|a_p| \ll 1$

Ramanujan $\Delta(z) = q\prod_{n\geq 1}(1-q^n)^{24} = q - 24q^2 + \dots = \sum_{n\geq 1} \tau(n) q^n$, cuspidal.

Conj: $|\tau(p)| \leq 2\cdot p^{11/2} = 2\cdot p^{\frac{12}{2}-\frac{1}{2}}$. Δ is wt 12. Deligne.

$|\tau(n)| \leq d(n)\cdot n^{11/2}$ "trivial".

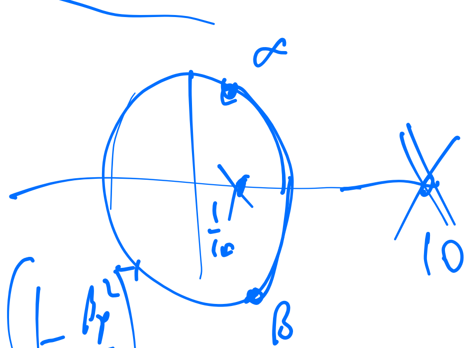
"Langlands-Satake params"

degree 2 L-function $L(\varphi, s) = \prod_p \left(1-\frac{\alpha_p}{p^s}\right)^{-1}\left(1-\frac{\beta_p}{p^s}\right)^{-1}$

φ is Hecke eigenform

$= \prod_p \left(1+\frac{a_p}{p^s}+\frac{a_{p^2}}{p^{2s}}+\dots\right)$ $\left(1-\frac{\alpha_p+\beta_p}{p^s}+\frac{\alpha_p\cdot\beta_p}{p^{2s}}\right)^{-1}$

$\mathbb{R} \ni a_p = \alpha_p + \beta_p$, $\alpha_p \cdot \beta_p = 1$.



$|a_p| \le 2$.

$$\prod_p \left(1-\frac{\alpha_p^2}{p^s}\right)^{-1}\left(1-\frac{\alpha_p\beta_p}{p^s}\right)^{-1}\left(1-\frac{\beta_p^2}{p^s}\right)^{-1}$$

Rankin-Selberg: $\sum \frac{a_n^2}{n^s}$ is an L-function has FE & Euler product.

$|a_n| < n^{1/2}$ $\Downarrow$ $|a_n| < n^{1/4}$

Hope: $\cancel{\sum \frac{a_n^3}{n^s}}$. $\Downarrow$ $\boxed{|a_n| < n^{1/6}}$, Not have analytic cont.

"Right object": $L(\mathrm{sym}^3\pi, s) := \prod_p \left(1-\frac{\alpha_p^3}{p^s}\right)^{-1}\left(1-\frac{\alpha_p^2\beta_p}{p^s}\right)^{-1}\left(1-\frac{\alpha_p\beta_p^2}{p^s}\right)^{-1}\left(1-\frac{\beta_p^3}{p^s}\right)^{-1}$

Shahidi

Kim-Shahidi

Kim-Sarnak $|a_p| \le 2\cdot p^{7/64}$.

Serre: if $\mathrm{sym}^n\pi$ rep is aut $\Rightarrow$ Ramanujan.

$|X| \le 1$, $\Leftarrow$ $|X^n| \le 10$. $\forall n$,

$T^1\mathbb{H} \simeq PSL_2(\mathbb{R})$, $(z, \zeta) = g\cdot(i, \uparrow) \leftrightarrow g$

(invariant) Haar measure $dg = dx\,\frac{dy}{y^2}\,d\theta$.

$g = n_x\, a_y\, k_\theta$ (Iwasawa decomp)

$\begin{pmatrix}1&x\\0&1\end{pmatrix}\begin{pmatrix}\sqrt{y}&0\\0&1/\sqrt{y}\end{pmatrix}\begin{pmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{pmatrix}$.

Def'n If $f:\mathbb{H}\to\mathbb{C}$, hol'c, $f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right) = (cz+d)^{k} f(z)$ $\forall \gamma\in\Gamma$, $z \in \mathbb{H}$,

& $f \in L^2$, $\langle f, f\rangle = \int f\cdot\bar{f}\, y^k \frac{dx\,dy}{y^2} < \infty$.

Then f is a cuspidal mod form $\Gamma\backslash\mathbb{H}$, D.

D.

$$f(z+1) = \sum_{n\in\mathbb{Z}} \underbrace{a_n\, e^{-2\pi n y}}\cdot e^{2\pi i n x}$$

$$\infty > \langle f, f\rangle \geq \int_{1_0}^{\infty}\left[\int_{-1/2}^{1/2} |f|^2\, dx\right] y^k \frac{dy}{y^2}.$$

Parseval:

$$\infty > \int_{1_0}^{\infty} \int_{\mathbb{R}/\mathbb{Z}} |f(x+iy)|^2\, dx\, y^k \frac{dy}{y^2} = \sum_{n\in\mathbb{Z}} |a_n|^2 \underbrace{\int_{1_0}^{\infty} e^{-4\pi n y}\, y^k \frac{dy}{y^2}}$$

diverges unless $n \geq 1$.

$\Rightarrow a_n = 0$ unless $n \geq 1$.

$$\Rightarrow f(z) = \sum_{n\geq 1} a_n\cdot q^n.$$

If f has at most poly growth as $y \to \infty$ ("moderate"), then f is modular. (not nec. cuspidal).

$\Updownarrow$ $a_n = 0$ unless $n \geq 0$. ("Eisenstein series").

$\boxed{\text{moderate growth} \Rightarrow \text{bdd.}}$

<u>Recall</u>: $\mathbb{A} = \mathbb{R} \times \prod_p{}' \mathbb{Q}_p = \{(a_\infty, a_2, \ldots),\ a_p \in \mathbb{Z}_p \text{ a.a. } p\}$

$\mathbb{Q}$ discrete in $\mathbb{A}$, Fund Dom $\mathbb{Q} \backslash \mathbb{A} = (0,1) \times \prod_p \mathbb{Z}_p$.

Fund Dom for $\mathbb{Q}^\times \backslash \mathbb{A}^\times = \underbrace{(0,\infty)}_{\mathbb{Z}^\times \backslash \mathbb{R}^\times} \times \prod_p \mathbb{Z}_p^\times$.

<u>Thm ($GL_2(\mathbb{R})$ <u>Iwasawa</u> decomp)</u>: $\forall g \in GL_2(\mathbb{R})$,

$\exists!$ decomp: $g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}}_{SO(2)} \underbrace{\begin{pmatrix} \pm 1 & \\ & 1 \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}}$

$r, y > 0$, $x \in \mathbb{R}$, $\theta \in \mathbb{R}/2\pi\mathbb{Z}$.

$$GL_2(\mathbb{Q}_p) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ a,b,c,d \in \mathbb{Q}_p, \ |ad-bc|_p \neq 0 \right\}.$$

$\Updownarrow$ $\in \mathbb{Q}_p^\times$.

inverse: $\frac{1}{\boxed{ad-bc}} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

max cpt $K_p = GL_2(\mathbb{Z}_p) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ a,b,c,d \in \mathbb{Z}_p, \ |\underbrace{ad-bc}_{\in \mathbb{Z}_p^\times}|_p = 1 \right\}$

Thm (Iwasawa decomp): $\forall g \in GL_2(\mathbb{Q}_p)$,

$\exists!$ decomp: $g = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^{e_1} & 0 \\ 0 & p^{e_2} \end{pmatrix} \cdot k$, ($u \in \mathbb{Q}_p$)

$k \in K_p$, $e_1, e_2 \in \mathbb{Z}$, & u

pf: $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} p^\ell d & \beta \\ -cp^\ell & \alpha \end{pmatrix} = \begin{pmatrix} & \\ 0 & \end{pmatrix}$

Let $p^\ell = \max(|c|_p, |d|_p)$ $\quad p^\ell c = c_{-2} p^{\ell} + \cdots +$

$p^\ell d = \ldots$

$p^3 \quad d_{-3} p^{-3} + \ldots$

Then $\max(|p^e d|_p, |p^e c|_p) = 1$. & at least one is a unit. Need to solve: $|\underbrace{p^e d}_{\in \mathbb{Z}_p^\times} \cdot \underbrace{\alpha} + \beta \cdot c \underbrace{p^e}|_p = 1$.

Find α & β ✓.

$$g \cdot k = \begin{pmatrix} t_1 & x \\ 0 & t_2 \end{pmatrix}$$

$t_1, t_2 \in \mathbb{Q}_p$.

$t_1 = p^{e_1} \cdot \underbrace{u_1}_{\in \mathbb{Z}_p^\times}$, $t_2 = p^{e_2} \cdot u_2$.

$$\underbrace{g \cdot k \begin{pmatrix} u_1^{-1} & \\ & u_2^{-1} \end{pmatrix}}_{\in K} \begin{pmatrix} p^{-e_1} & \\ & p^{-e_2} \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \quad x \in \mathbb{Q}_p,$$

$$\Rightarrow g = \begin{pmatrix} 1 & \boxed{x} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p^{e_1} & \\ & p^{e_2} \end{pmatrix} \cdot k.$$

Uniqueness? <u>No</u>. ~~Hmk:~~ <u>How to fix?</u> ...