Last time: $G = SL_2(\mathbb{R}) \curvearrowright \mathbb{H} \cong SL_2(\mathbb{R})/SO(2)$.

$\checkmark$

$SL_2(\mathbb{Z}) = \left\langle \begin{pmatrix}1&1\\0&1\end{pmatrix}, \begin{pmatrix}0&1\\-1&0\end{pmatrix} \right\rangle$,

Classification
- elliptic $|tr|<2$
- hyperbolic $|tr|>2$
- parabolic/unipotent $tr=\pm 2$

Possibilities for fixed pts of $g \in G$

$\begin{pmatrix}a&b\\c&d\end{pmatrix} \circ z = z$, $\Rightarrow az+b = cz^2+dz$. $\Rightarrow z = \dfrac{-(d-a) \pm \sqrt{(d-a)^2+4bc}}{2c}$

" $\dfrac{az+b}{cz+d}$ $\Rightarrow \underbrace{cz^2+(d-a)z-b=0}$.

$\Rightarrow \underbrace{z_+, z_- \in \mathbb{R}, \neq}$, or $z_+ = z_- \in \mathbb{R}$, or $z_+ \in \mathbb{H}$, $z_- \in \mathbb{H}^-$.

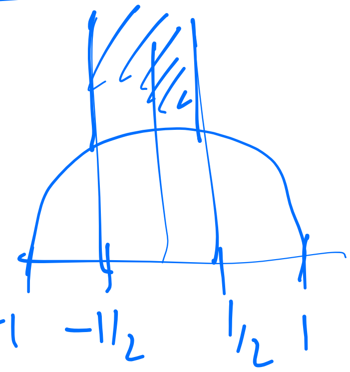
$\Leftrightarrow (d-a)^2+4bc = d^2-2ad+a^2+4bc = d^2+2ad+a^2 - 4(\cancel{ad-bc})^{1}$.

$= (tr)^2 - 4$. $\begin{cases} >0 \Rightarrow \text{hyp} \\ =0 \Rightarrow \text{parabolic} \\ <0 \Rightarrow \text{elliptic.} \end{cases}$

$\boxed{\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}z\\1\end{pmatrix} = \lambda \begin{pmatrix}z\\1\end{pmatrix} \sim \begin{pmatrix}z\\1\end{pmatrix}}$

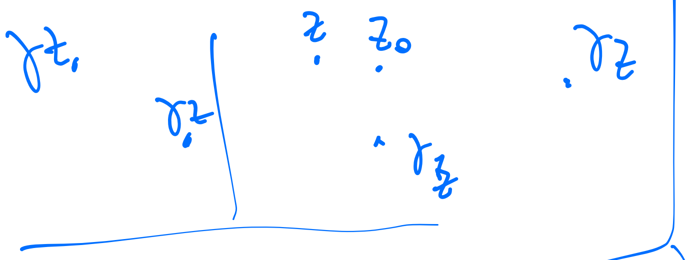
x^2+Dy^2.

Find dom D for $\underbrace{(P)SL_2(\mathbb{Z})}_{=\Gamma} \curvearrowright \mathbb{H}$

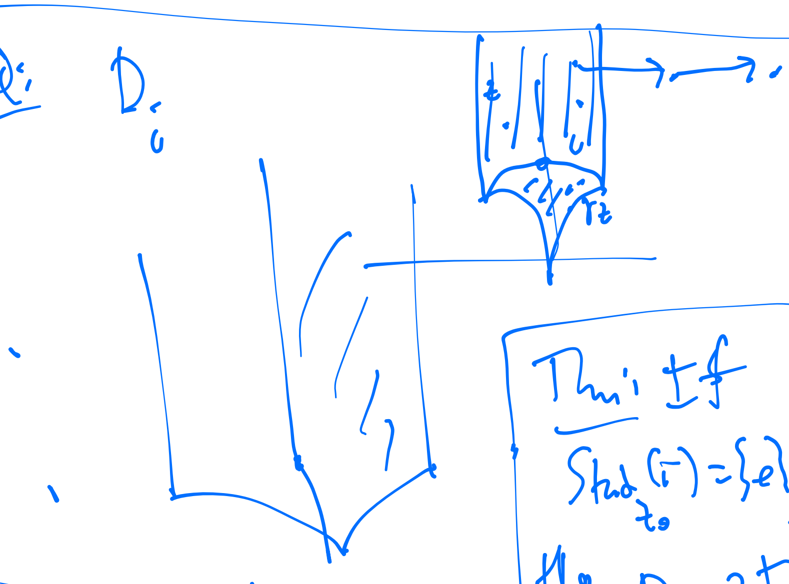


Def. Given any $z_0 \in \mathbb{H}$ s.t. $\underbrace{Stab_{z_0}(\Gamma) = \{e\}}$, a Dirichlet domain for Γ (based at z_0) is:

$$D_{z_0} := \{ z \in \mathbb{H} : \forall \gamma \in \Gamma \setminus e,\ d(z, z_0) \le \underbrace{d(\gamma z, z_0)}^{d(z, \gamma^{-1} z_0)} \}.$$

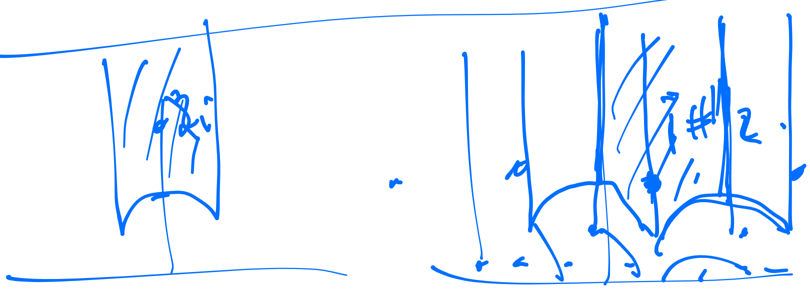


Q: D_i



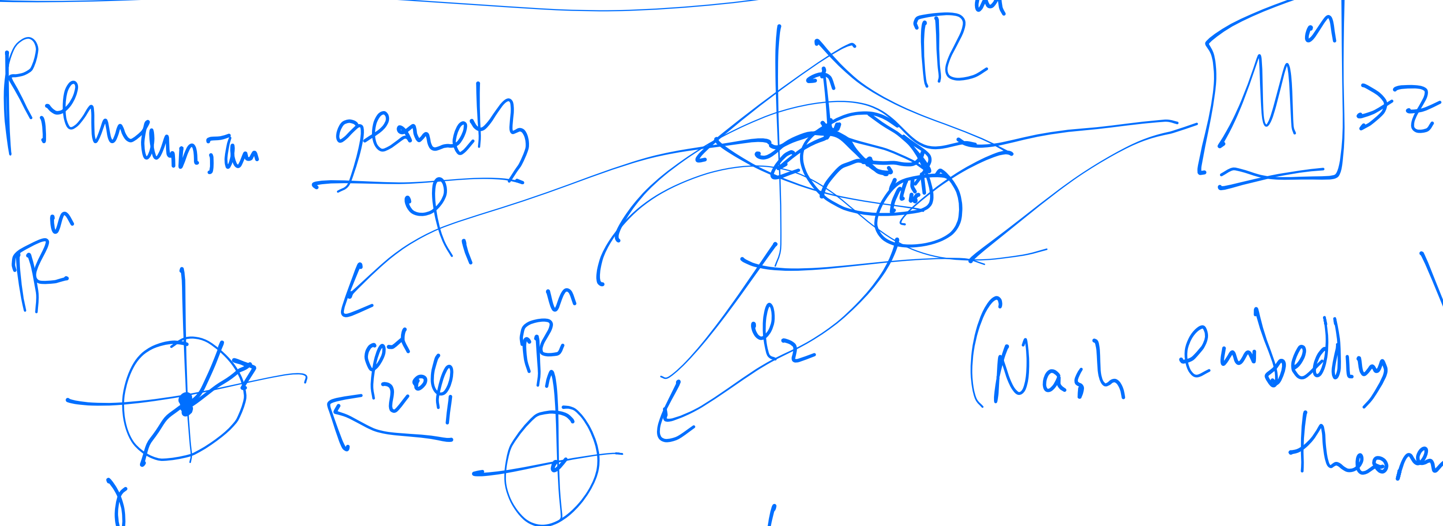
Voronoi cells

This: If $\mathrm{Stab}_{z_0}(\Gamma) = \{e\}$, then D_{z_0} is FD.



How to do geometry on $\mathbb{H}$?

Riemannian geometry



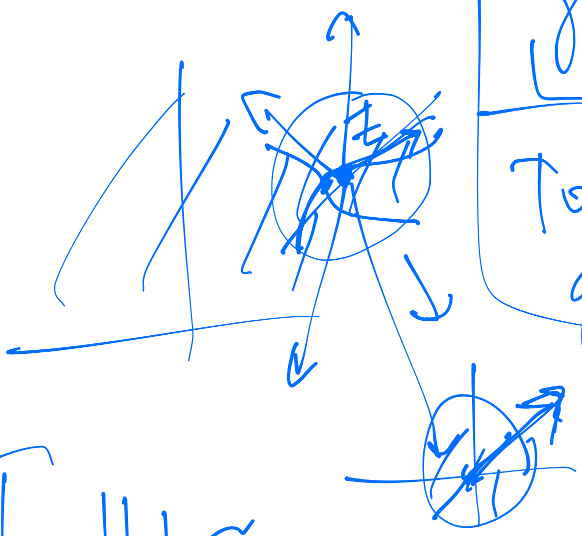
(Nash embedding theorem)

$$T_z M = \{ \gamma : [-1, 1] \xrightarrow{t} M,\ \gamma(0) = z,$$

$\{\gamma'(0)$ exists$\} / \sim$

$\gamma_1 \sim \gamma_2 \iff \gamma_1'(0) = \gamma_2'(0)$.

$\mathbb{H}$

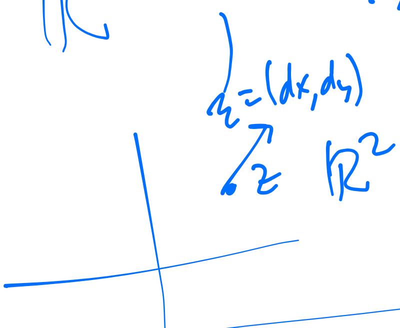


To do geom: quad form on T_zM,

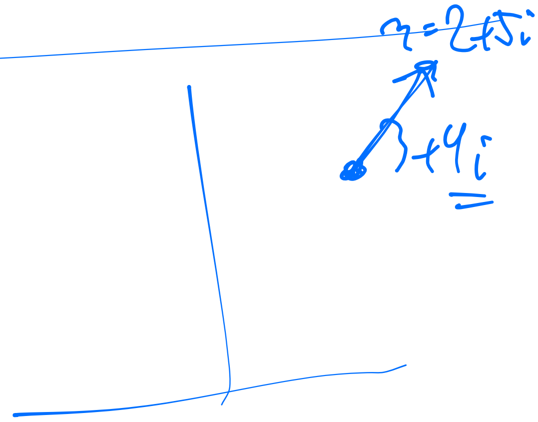
i.e. to each $\eta \in T_z\mathbb{H}$.

$T_z\mathbb{H} \cong \mathbb{C}$, want $q_z(\eta)$ = length of η.

$\mathbb{R}^2$ $\quad \eta \in T_z\mathbb{R}^2$, $\quad q_z(\eta) = dx^2 + dy^2$ = Euclidean quad form,



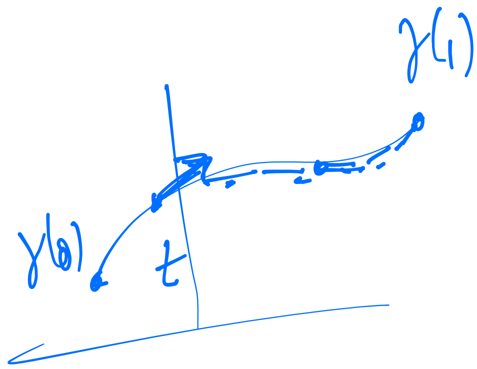
On $\mathbb{H}$: $\quad q_z(\eta) = \dfrac{dx^2 + dy^2}{\operatorname{Im} z^2}$

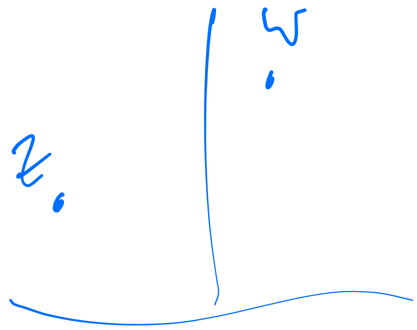


$\|\eta\|^2 = \dfrac{4+25}{4}$

Given $\gamma : [0,1] \to \mathbb{H}$,

$\ell(\gamma) = \int_0^1 \dfrac{|\gamma'(t)|}{\operatorname{Im}\gamma(t)}\, dt$.





Geodesics in $\mathbb{H}$?

curves of minimal length between z & w?

$\gamma(0)=z$, $\gamma(1)=w$, $\gamma(t)=x(t)+iy(t)$

minimize $\displaystyle\int_0^1 \frac{\sqrt{\dot{x}(t)^2+\dot{y}(t)^2}}{y(t)}\,dt$.

$\displaystyle\geq \int_0^1 \frac{|\dot{y}|}{y}\,dt$.

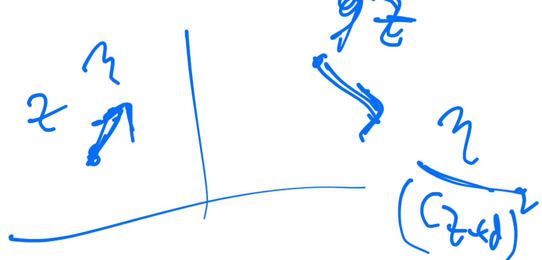
So minimal curve is vertical

minimized when curve doesn't change sign of $\dot{y}$.

$\boxed{\text{geod if } t_{int}=t_{fin} \text{ is straight vertical line}}$

$g\in SL_2(\mathbb{R}) \curvearrowright T\mathbb{H} = \bigsqcup T_z\mathbb{H}$,

$$g\cdot(z,\eta) = \left(gz, \frac{\eta}{(cz+d)^2}\right).$$



$$D\left(\frac{az+d}{cz+d}\right) = \frac{(cz+d)(a)-(az+d)c}{(cz+d)^2} = \frac{1}{(cz+d)^2}$$

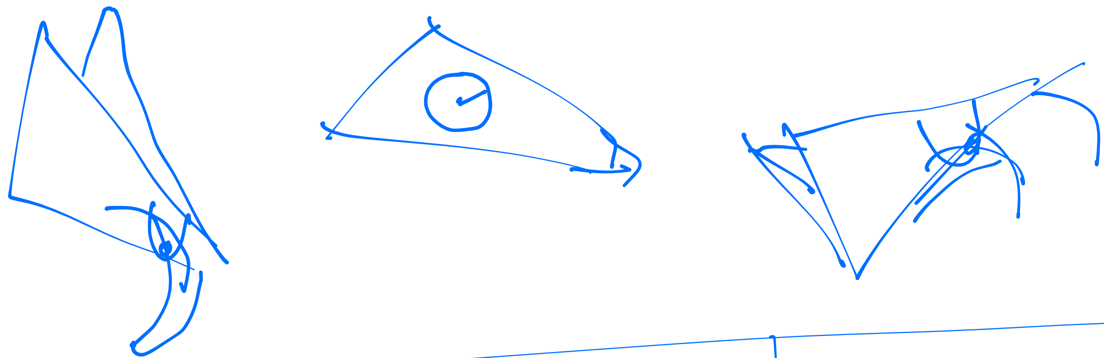
↳ action of $SL_2(\mathbb{R})$ is isometric!

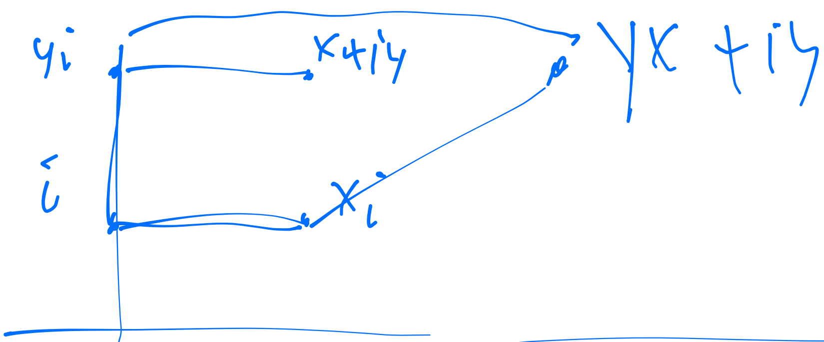
$$\left\| \frac{\eta}{(cz+d)^2} \right\|_{gz}$$

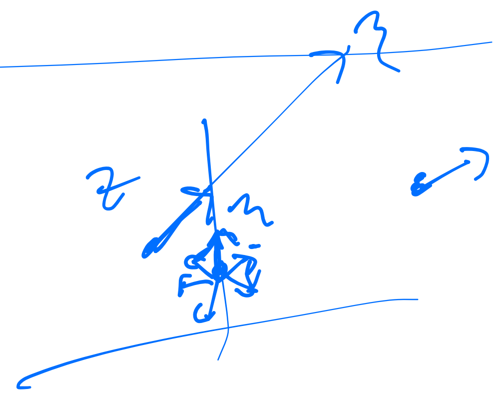
$$\boxed{\|\eta\|_z = \frac{|\eta|}{\operatorname{Im} z}}$$

$$\frac{\left| \frac{\eta}{(cz+d)^2} \right|}{\operatorname{Im} gz} = \frac{|\eta|}{\cancel{|(cz+d)^2|}} \cdot \frac{\cancel{|cz+d|^2}}{\operatorname{Im} z}$$

$$\boxed{\operatorname{Im} gz = \frac{\operatorname{Im} z}{|cz+d|^2}}$$





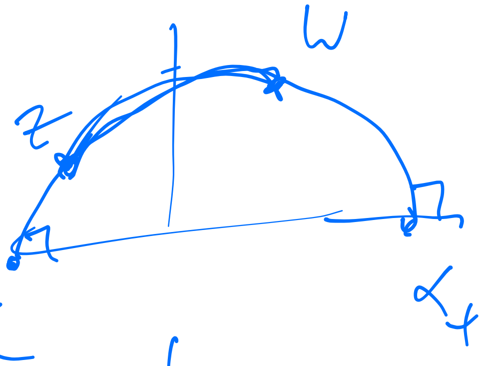


Unit tangent bundle $T^1\mathbb{H} = \{(z,\eta) \in T\mathbb{H}, \|\eta\|_z = 1\}$

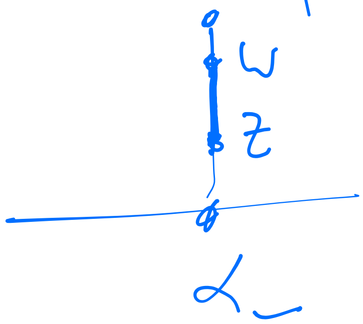
Fact: (1) $SL_2(\mathbb{R}) \simeq T^1\mathbb{H}$.

$$\underbrace{SO(2)}\cdot i = i$$

If $z, w \in \mathbb{H}$, $\exists$ circle through z & w & $\perp$ to $\mathbb{R}$.



$\downarrow$



Exercise: $\exists!\ g \in SL_2(\mathbb{R})$

$$g\begin{pmatrix}\alpha_- \\ \alpha_+ \\ z\end{pmatrix} = \begin{pmatrix}0 \\ i\infty \\ i\end{pmatrix}$$

Invariant measure on $\mathbb{H}$ & $T^1\mathbb{H}$,

$$\boxed{d\mu(gz) = d\mu(z)}.$$

$$\int_{\mathbb{H}} f\, d\mu(z) = \int f(gz)\, d\mu(gz)$$

$z = x+iy$, $\quad d\mu(z) = \dfrac{dx\,dy}{y^2}$ is invariant under $SL_2(\mathbb{R})$

$SL_2(\mathbb{R}) = G = NAK$ Iwasawa decomposition.

$$g \in G$$

$$g = \underbrace{\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}\sqrt{y} & \\ & \frac{1}{\sqrt{y}}\end{pmatrix}}\underbrace{\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}} = n\cdot a\cdot k.$$

$$\begin{pmatrix}a & b\\ c & d\end{pmatrix}\underbrace{\begin{pmatrix}\frac{d}{\sqrt{d^2+c^2}} & \frac{c}{\delta}\\ \frac{-c}{\delta} & \frac{d}{\delta}\end{pmatrix}} = \begin{pmatrix}\sqrt{y} & *\\ 0 & \frac{1}{\sqrt{y}}\end{pmatrix}$$

$$z \mapsto \begin{pmatrix}\sqrt{\alpha} & 0\\ 0 & \frac{1}{\sqrt{\alpha}}\end{pmatrix}\circ z = \alpha\cdot z, \quad x \mapsto \alpha x,$$
$$y \mapsto \alpha y.$$

$$\frac{dx\,dy}{y^2} \mapsto \frac{\alpha\cdot dx\,\alpha\cdot dy}{\alpha^2 y^2} = \frac{dx\,dy}{y^2}.$$

Invariant meas on $\underline{T^1\mathbb{H}} = \{(z,\zeta) \mid \|\zeta\|_z = 1\}$

$$= \underbrace{SL_2(\mathbb{R})}$$

invariant Haar measure

$$\underbrace{dg} = dn\cdot da\cdot dk$$
$$= \underbrace{dx\,\frac{dy}{y^2}\cdot d\theta}.$$

$L^2\left(SL_2(\mathbb{Z})\backslash\mathbb{H}, \frac{dxdy}{y^2}\right).$

$\left\{ f : \mathbb{H} \to \mathbb{C},\ f(\gamma z) = f(z),\ \int_D |f(z)|^2 \frac{dxdy}{y^2} < \infty \right\},$

$\Delta = y^2 \cdot \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$ Laplace operator.

- Maass form: $\varphi \in L^2(SL_2(\mathbb{Z})\backslash\mathbb{H})$ eigenfunction of Δ.

  & φ bounded at ∞.

- Modular form $f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} z\right) = (cz+d)^k f(z)$.

$\left| f \cdot (\operatorname{Im} z)^{k/2} \right|$ is automorphic.

automorphic ↓

Petersson inner product $\langle f, g \rangle = \int_{SL_2(\mathbb{Z})\backslash\mathbb{H}} \boxed{f(z)\, \overline{g(z)}\, y^k} \frac{dxdy}{y^2}$

$L^2(SL_2(\mathbb{Z})\backslash G, dg)^K \quad G \curvearrowleft \quad SL_2(\mathbb{Z})\backslash\mathbb{H}$