<u>Last time:</u> Given local character $\omega_p : \mathbb{Q}_p^\times \to \mathbb{C}^\times$,

$f_p \in S(\mathbb{Q}_p)$, $\quad Z_p(\omega_p, f_p, s) = L_p(\omega_p, f_p, s)$.

$$= \int_{\mathbb{Q}_p^\times} \omega_p(t_p)\, f_p(t_p)\, |t_p|_p^{s}\, d^\times t_p .$$

(<u>Recall:</u> If ω_p unramified, & $f_p = \mathbb{1}_{\mathbb{Z}_p}$, then

$$Z_p(\omega_p, f_p, s) \underset{(\operatorname{Re} s > 0)}{=} L_p(\chi, s) = \frac{1}{1 - \frac{\chi(p)}{p^s}} .\text{)}$$

<u>Lemma:</u> $Z_p(\omega_p, f_p, s)$ Conv abs in $\underline{\operatorname{Re} s > 0}$.

<u>pf:</u> $$|Z_p(\omega_p, f_p, s)| \le \int_{p^{-N}\mathbb{Z}_p} \cancel{|f_p|}\ \cancel{|\omega_p|}^{1}\ |t_p|_p^{\operatorname{Re} s}\, d^\times t_p$$

$$\le C \cdot \sum_{n=-N}^{\infty} \int_{t_p \in p^n \mathbb{Z}_p^\times} p^{-n\,\mathrm{Re}\, s} \, d^\times t_p,$$

$$= \text{geom series in } \sum_{n \geq 0}^{\infty} \left(p^{-\mathrm{Re}(s)}\right)^n + \text{finite.}$$

Lemma: $\exists$ mer'c $\gamma_{\omega_p}(s)$ s.t.

$$Z_p(\omega_p, f_p, s) = \underbrace{\gamma_{\omega_p}(s)}_{\text{indep of } f_p} \, Z_p(\bar{\omega}_p, \hat{f}_p, 1-s).$$

Notice: indep of f_p.

Pf: In $0 < \mathrm{Re}\, s < 1$, want:

$$\frac{Z_p(\ldots g)}{Z_p(\hat{g})} \overset{?}{=} \frac{Z_p(\omega_p, f_p, s)}{Z_p(\bar{\omega}_p, \hat{f}_p, 1-s)} \text{ indep of } f.$$

For another $g_p \in S(\mathbb{Q}_p)$.

Look at $Z_p(\omega_p, f_p, s)\cdot Z_p(\bar{\omega}_p, \hat{g}_p, 1-s)$.

$$= \int_{\mathbb{Q}_p^\times} \omega_p(t_p)\, f_p(t_p)\, |t_p|_p^{s}\, d^\times t_p$$

$$\times \int_{\mathbb{Q}_p^\times} \bar{\omega}_p(u_p) \left[\int_{\mathbb{Q}_p \setminus \{0\}} g_p(v_p)\, e^{2\pi i v_p u_p}\, dv_p \right] |u_p|_p^{1-s}\, d^\times u_p$$

$$u_p \mapsto u_p \cdot t_p.$$

$$= \int_{\mathbb{Q}_p^\times} \cancel{\omega_p(t_p)}\, f_p(t_p)\, \cancel{|t_p|_p^{s}} \int_{\mathbb{Q}_p^\times} \bar{\omega}_p(u_p)\, \cancel{\bar{\omega}_p(t_p)} \int_{\mathbb{Q}_p^\times} g_p(v_p)\, e^{2\pi i v_p u_p t_p}$$

$$dv_p\, |u_p|_p^{1-s}\, \cancel{|t_p|^{1-s}}\, d^\times u_p\, \frac{dt_p \cdot p}{\cancel{|t_p|}\,(p-1)}$$

$$= \frac{p}{p-1} \int_{\mathbb{Q}_p^\times} \int_{\mathbb{Q}_p^\times} \int_{\mathbb{Q}_p^\times} \underbrace{f_p(t_p)} \underbrace{g_p(v_p)} e^{2\pi i v_p u_p t_p} |u_p|_p^{1-s} \overline{\omega_p(u_p)} \, d^\times u_p \, dv_p \, dt_p$$

symmetric. ✓.

Let's find γ_ω's. $\begin{cases} 1 & \chi(-1)=1 \leftarrow \text{unramified} \\ \mathrm{sgn} & \chi(-1)=-1 \leftarrow \text{ramified} \end{cases}$

Case 1: $p=\infty$. $Z_\infty(\omega_\infty, f_\infty, s)$ $\hat{f}_\infty = f_\infty$.

unramified $\Rightarrow$ choose $f_\infty(x) = e^{-\pi x^2}$.

$$= \int_{\mathbb{R}^\times} e^{-\pi x^2} |x|^s \frac{dx}{|x|} = 2\pi^{-s/2}\Gamma\left(\frac{s}{2}\right).$$

$$\Rightarrow \gamma_{\omega_\infty}(s) = \frac{\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)}{\pi^{-\frac{(1-s)}{2}}\Gamma\left(\frac{1-s}{2}\right)} \text{ mer'c in } \mathbb{C},$$

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ramified $\Rightarrow$ choose $f_\infty(x) = x e^{-\pi x^2}$, $\hat{f}_\infty = i f_\infty$.

$$\Rightarrow Z_p = \int_{\mathbb{R}^\times} \operatorname{sgn} x \cdot x\, e^{-\pi x^2} |x|^s \frac{dx}{|x|}$$

$$= 2\, \pi^{-\frac{(s+1)}{2}} \Gamma\left(\frac{s+1}{2}\right).$$

$$\gamma_{\omega_\infty}(s) = \frac{\pi^{-\frac{(s+1)}{2}} \Gamma\left(\frac{s+1}{2}\right)}{i\, \pi^{-\frac{(1-s+1)}{2}} \Gamma\left(\frac{1-s+1}{2}\right)}.$$

also mer'c.

Case 2: p unramified, $f_p = \mathbb{1}_{\mathbb{Z}_p}$.

$$Z_p = \frac{1}{1 - \frac{\chi(p)}{p^s}}, \qquad \gamma_{\omega_p}(s) = \frac{1 - \frac{\overline{\chi}(p)}{p^{1-s}}}{1 - \frac{\chi(p)}{p^s}}.$$

Case 3: p ramified, $\ker \omega_p \supset 1 + p^\ell \mathbb{Z}_p$, w/ ℓ minimal. & $\omega_p\left(p^n \left(\underset{\in \mathbb{Z}_p}{m + p^\ell u}\right)\right) = \overline{\chi(m)}$, $m \in (\mathbb{Z}/p^\ell\mathbb{Z})^\times$

Choose $\boxed{f_p = \mathbb{1}_{1 + p^\ell \mathbb{Z}_p}}$

$$Z_p(\omega_p, f_p, s) = \int_{|t_p|^{\ell}_{\mathbb{Z}_p} \cancel{\mathbb{Q}_p^\times}} \cancel{\omega_p(x_p)}^{1} \cancel{f_p(t_p)}^{1} \cancel{|t_p|_p}^{s} \cancel{d^\times t_p} \quad \frac{dt_p}{\cancel{|t_p|_p}} \cdot \frac{p}{p-1}$$

$$\boxed{\hat{f}_p(\xi) = e^{2\pi i \xi} p^{-\ell} \cdot \mathbb{1}_{p^{-\ell}\mathbb{Z}_p}}$$

$$= \frac{p}{p-1} p^{-\ell}.$$

$$Z_p(\bar{\omega}_p, \hat{f}_p, 1-s) = \int_{\mathbb{Q}_p^\times} \bar{\omega}_p(t_p)\, e^{2\pi i t_p}\, p^{-\ell}\, \mathbb{1}_{p^{-\ell}\mathbb{Z}_p}(t)\, |t_p|_p^{1-s}\, \cancel{d^\times t_p} \quad \frac{dt_p}{\cancel{|t_p|}} \cdot \frac{p}{(p-1)}$$

$$\ell \geq n$$

$t_p \in p^{-\ell}\mathbb{Z}_p$, write $t_p = p^{-n}(m + p^\ell u)$

$$\underset{m \in (\mathbb{Z}/p^\ell\mathbb{Z})^\times}{\hat{}} \qquad \underset{u \in \mathbb{Z}_p}{\hat{}}$$

$a_{-\ell}p^{-\ell} + \dots + a_{-1}p^{-1} + a_0 + \dots$

$dt_p = p^{\ell-n}\, du$

$$\to = \frac{p^{-\ell} \cdot p}{(p-1)} \sum_{n=1}^{\ell} \sum_{m \bmod p^\ell}' \chi(m) \int_{u \in \mathbb{Z}_p} e^{2\pi i \frac{m}{p^n}} p^{-ns} p^{\ell-n}\, du + p \sum_{\cancel{n=0}}^{p^{\ell-n}} \sum_{m \in (\mathbb{Z}/p^\ell)}' \chi(m) \int_{u \in \mathbb{Z}_p} \cancel{1\, p^{-ns}}^{0}\, du$$

$$= \frac{p}{p-1}\, \cancel{p^{-\ell}} \sum_{n=1}^{\ell} \underbrace{\sum_{m \bmod p^\ell}{}' \chi(m)\, e^{2\pi i \frac{m}{p^n}}}\; \underbrace{p^{\cancel{\ell}-n}\; p^{-ns}}.$$

Note: $$\sum_{m \bmod p^\ell}{}' \chi(m)\, e^{2\pi i \frac{m}{p^n}} = \begin{cases} 0, & n<\ell \\ \tau(\chi), & \underline{n=\ell} \end{cases}$$

$$= \frac{p}{p-1}\, p^{-\ell(1+s)}\, \tau(\chi).$$

If p ramified with conductor $\underline{\underline{p^\ell}}$, then

$$\gamma_{\omega_p}(s) = \frac{\cancel{\frac{p}{(p-1)}}\;\cancel{p^{-\ell}}}{\cancel{\frac{p}{p-1}}\; p^{-\ell(1+s)}\, \tau(\chi)} = \frac{p^{\ell s}}{\tau(\chi)}.$$ $\underline{\text{nice.}}$

Lemma: $\gamma_{\omega_p}(s)\cdot\gamma_{\bar\omega_p}(1-s) = \underbrace{\chi(-1)}$.

pf: $Z_p(\omega_p, f_p, s) = \underbrace{\gamma_{\omega_p}(s)\, Z_p(\bar\omega_p, \hat f_p, 1-s)}$,

$$Z_p(\bar\omega_p, \hat f_p, 1-s) = \gamma_{\bar\omega_p}(1-s)\cdot Z_p(\omega_p, \hat{\hat f}_p, s),$$

$$\boxed{\hat{\hat f}_p(x) = f_p(-x),}$$

$$= \gamma_{\bar\omega_p}(1-s)\, Z_p(\omega_p(-\cdot), f_p, s),$$

$$= \chi(-1)\,\gamma_{\bar\omega_p}(1-s)\, \underbrace{Z_p(\omega, f, s)}.$$

$$= \underbrace{\chi(-1)\,\gamma_{\bar\omega_p}(1-s)\,\gamma_{\omega_p}(s)}_{\text{“}1\text{”}}\, Z_p(\bar\omega_p, \hat f, 1-s).$$

Test when p = ramified. $\boxed{\tau(\chi) = \sum'_{m(p^e)} \chi(m)\, e^{2\pi i \frac{m}{p^e}}}$

$$\gamma_{\omega_p}(s) = \frac{p^{\ell s}}{\tau(\chi)}, \quad \gamma_{\omega_p}(s)\cdot\gamma_{\overline{\omega}_p}(1-s) = \chi(-1),$$

$$= \frac{\cancel{p^{\ell s}}}{\tau(\chi)} \cdot \frac{p^{\ell(1-\cancel{s})}}{\tau(\overline{\chi})}.$$

$$\boxed{\text{Fact: } \tau(\chi) = \begin{cases} p^{\ell/2}, & \chi(-1)=1 \\ i\,p^{\ell/2}, & \chi(-1)=-1 \end{cases}}$$

$$= \frac{\cancel{p^{\ell}}}{\cancel{p^{\ell}}} \begin{cases} 1, & \chi(-1)=1 \\ -1, & \chi(-1)=-1. \end{cases} = \chi(-1).$$

"Rankin–Selberg <u>convolution</u>."

$$L_\alpha(s) = \prod_{n=1}^{d} \prod_p \left(1 - \frac{\alpha_n(p)}{p^s}\right)^{-1}$$

Langlands–Satake params.

$$L_\beta(s) = \prod_{m=1}^{e} \prod_p \left(1 - \frac{\beta_m(p)}{p^s}\right)^{-1}.$$

$$L_{\alpha\times\beta}(s) = \prod_{n=1}^{d} \prod_{m=1}^{e} \prod_{p} \left(1 - \frac{\alpha_n(p)\beta_m(p)}{p^s}\right)^{-1}.$$

For $GL(1,\mathbb{A})$, ϕ_1, ϕ_2 aut reps.

$\phi_1 \rightsquigarrow \chi_1$, $\phi_2 \to \chi_2$, $L(\phi_i, s) = \prod_p \left(1 - \frac{\chi_i(p)}{p^s}\right)^{-1}$

$$L(\phi_1 \times \phi_2, s) = \prod_p \left(1 - \frac{\chi_1(p)\chi_2(p)}{p^s}\right)^{-1}.$$

For $f \in S(\mathbb{A})$, $\vartheta_f(t) = \sum_{q \in \mathbb{Q}^\times} f(qt)$.

Look at:

$$\int_{\mathbb{Q}^\times \backslash \mathbb{A}^\times} \vartheta_f(t)\, \phi_1(t)\, \phi_2(t)\, |t|_{\mathbb{A}}^s \, d^\times t.$$

$$= \int_{\mathbb{A}^\times} f(t)\, \phi_1(t)\, \phi_2(t)\, |t|_{\mathbb{A}}^s \, d^\times t.$$

Same calc as always leads to $L(\varphi_1 * \varphi_2, s)$.

Why "convolution"?

$$\widehat{f \cdot g} = \hat{f} * \hat{g}$$

$$\widehat{\hat{f} \cdot \hat{g}}^{-1} = f * g.$$

$$L_a(s) = \sum \frac{a_n}{n^s}, \quad L_b(s) = \sum \frac{b_n}{n^s}.$$

$$L_{a \times b}(s) = \sum \frac{a_n \cdot b_n}{n^s}.$$

$$L_a(s) \cdot L_b(s) = \sum \frac{1}{n^s} \left(\sum_{d \mid n} a_d \, b_{n/d} . \right)$$