Last time: Duality between primitive Dirichlet characters $\chi(n) = \prod \chi \pmod{p^e}$ $\longleftrightarrow$ $GL(1)$ adelic aut rep.

$$\phi = c \cdot |\cdot|_{\mathbb{A}}^{it} \cdot \underbrace{\omega}$$

$\leftarrow$ Hecke character $\underbrace{\mathbb{Q}^\times \backslash \mathbb{A}^\times} \to \mathbb{C}^\times$.

$$\omega(g_\infty, g_2, \ldots) = \prod_{p \le \infty} \omega_p(g_p)$$

with

$$\omega_\infty(g_\infty) = \begin{cases} 1, & \chi(-1) = 1 \\ \operatorname{sign} g_\infty, & \chi(-1) = -1 \end{cases}$$

$\nu \ne p$ (ω_ν $\underline{\text{unramified}}$) $\omega_\nu(\nu^n \cdot \underset{\hat{\mathbb{Z}}_\nu^\times}{u}) = \overline{\chi(\nu)}^n$.

If $\nu = p$, ω_p triv on $\underbrace{1 + p^e \mathbb{Z}_p}$, $\left[\begin{aligned} &\omega_p(p^n \cdot (a_0 + a_1 p + \cdots + a_{e-1} p^{e-1} + \cdots)) \\ &= \overline{\chi(a_0 + a_1 p + \cdots + a_{e-1} p^{e-1})} \end{aligned}\right.$.

Need to check that $\omega(\underset{\mathbb{Q}^\times}{q}) = 1$.

Suffices to check on $\overset{\text{prime}}{\ell} \ne p$, $\omega(\ell) \overset{?}{=} 1$ ✓, $\omega(1/\ell) \overset{?}{=} 1$ ✓

& $\underbrace{\omega(-1) \overset{?}{=} 1}$ ✓ $\omega(p) \overset{?}{=} 1$ ✓, $\omega(1/p) \overset{?}{=} 1$ ✓

Look at $\underbrace{\omega(-1) = \prod_{\nu \le \infty} \omega_\nu(-1)}$. $\left[\begin{aligned} &\text{For } \nu = \infty, \ \omega_\nu(-1) = \chi(-1). \\ &\text{For } \nu \ne p, \ \omega_\nu(-1) = 1. \\ &\text{For } \nu = p, \ \omega_p(-1) = \chi(-1). \end{aligned}\right]$

Check $\omega(p) = \prod_{\nu \le \infty} \omega_\nu(p)$.

$\underbrace{\omega_\infty(p) = 1, \ \omega_{\nu \ne p}(p : \mathbb{Q}_\nu) = 1, \ \omega_p(p) = 1.}$

For ℓ prime $\neq p$, $\omega(\ell) = \prod_{v \in \mathbb{Q}} \omega_v(\ell)$.

$\omega_\infty(\ell) = 1$, $\omega_{v \neq \ell, p}(\ell) = 1$, $\omega_\ell(\ell) = \chi(\ell)$,
$\ell \nwarrow$ unramified

$\omega_p(\ell) = \overline{\chi(\ell)}$

$\omega_p(1/\ell) = \overline{\chi(\bar{\ell})} = \chi(\ell)$, $\omega_\ell(1/\ell) = \chi(\ell)^{-1}$

Aside: Gives a "easy" method for determining whether $\chi \pmod{N}$ is $\underline{\text{primitive}}$. $= \prod \underbrace{\chi_i \pmod{p^e}}_{\text{prim.}}$

$N = \prod p^e$

How to tell if χ mod p^e is primitive?

ω_p triv on $1 + p^e \mathbb{Z}_p$, with e minimal.

χ mod p^e imprimitive $\iff$ $\chi(1 + a_{e-1} p^{e-1}) = 1$. ($a_{e-1} \in \mathbb{F}_p$)

if $\exists\, a_{e-1} \in \mathbb{F}_p$ s.t. $\chi(1 + a_{e-1} p^{e-1}) \neq 1 \Rightarrow \chi$ primitive!

Back to χ mod $N = 12 = 4 \cdot 3$.

$(\mathbb{Z}/12\mathbb{Z})^\times = \{1, 5, 7, 11\}$.

	1	5	7	11
		1(4) 2(3)	3(4) 1(3)	
$\chi_0^{(4)}\cdot\chi_0^{(3)} = \chi_0$	1	1	1	1
$\chi_0^{(3)}\chi_4^{(4)} = \chi_1$	1	1	-1	-1
$\chi_0^{(4)}\cdot\chi_5^{(3)} = \chi_2$	1	-1	1	-1
$\chi_4^{(4)}\cdot\chi_5^{(3)} = \chi_3$	1	-1	-1	1

unique primitive mod 12

χ mod 3

	1	2
$\chi_0^{(3)}$	1	1
$\chi_5^{(3)}$	1	-1

χ mod 4

	1	3
$\chi_0^{(4)}$	1	1
$\chi_4^{(4)}$	1	-1

imprim χ mod 3,

imprim!!! χ mod 4,

LMFDB: Brian Conrey enumeration. $\chi(N) = \prod \chi(p^e)$,

$(p\neq 2)$ $\chi_{p^e}^{(j)}(n) = e^{2\pi i \frac{a\cdot b}{\phi(p^e)}}$ $(j,n\in(\mathbb{Z}/p^e\mathbb{Z})^\times)$. Let g be least generator mod p^e.

" $\chi_{p^e}^{(n)}(j)$ " $\quad j = g^a, \quad n = g^b, \quad m = g^c$

Imprimitive if: $(a, \phi(p^e)) > 1$.

$$\chi_{p^e}^{(j)}(n\cdot m) = e^{2\pi i \frac{a(b+c)}{\phi(p^e)}} = \chi_{p^e}^{(j)}(n)\cdot \chi_{p^e}^{(j)}(m).$$

Last time: $f\in S(\mathbb{A})$, " $\prod f_p$ ", $\vartheta_f(t) = \sum_{q\in\mathbb{Q}^\times} f(qt)$ $(t \in \mathbb{A}^\times)$ $\left(= \sum_{n\in\mathbb{Z}\setminus\{0\}} f(\tfrac{n}{N}t)\right)$

adelic Poisson sum.

$$= \frac{1}{|t|_{\mathbb{A}}}\vartheta_{\hat f}\left(\frac{1}{t}\right) + \frac{\hat f(0)}{|t|} - f(0).$$

twisted Mellin transform:

$$\int_{\mathbb{Q}^\times\backslash\mathbb{A}^\times} \vartheta_f(t)\cdot\phi_s(t)\,d^\times t = \int_{\mathbb{Q}^\times\backslash\mathbb{A}^\times} \vartheta_f(t)\,\omega(t)\cdot|t|_{\mathbb{A}}^{s}\,d^\times t.$$

$$\Downarrow$$

$$= \underbrace{\qquad}\cdot L(\chi,s),$$

$$\prod_p \left(1-\frac{\chi(p)}{p^s}\right)^{-1}$$

$$(\mathrm{Re}\,s>1),\quad \int_{\mathbb{A}^\times} f(t)\,\omega(t)\,|t|_{\mathbb{A}}^{s}\,d^\times t.$$

$$= \prod_p \int_{\mathbb{Q}_p^\times} f_p(t_p)\,\omega_p(t_p)\,|t_p|_p^{s}\,d^\times t_p.$$

$$\mathbb{Q}^\times\backslash\mathbb{A}^\times = D^\times = (0,\infty)\times\prod \mathbb{Z}_p^\times$$

$$\int_{D^\times} = \underbrace{\int_{D^\times,\,|t|_{\mathbb{A}}\le 1}}_{} + \underbrace{\int_{D^\times,\,|t|_{\mathbb{A}}\ge 1}}_{}$$

$$\int_{|t|_{\mathbb{A}}\ge 1} \vartheta_f(t)\,\omega(t)\,|t|_{\mathbb{A}}^{s}\,d^\times t$$

entre in s.

$$\sum_{n\in\mathbb{Z}\setminus\{0\}} f\left(\frac{n}{N}t\right)$$

$$= \int_{|t|_{\mathbb{A}}\le 1} \left[\frac{1}{|t|_{\mathbb{A}}}\vartheta_{\hat f}\left(\frac{1}{t}\right)\right]\omega(t)\,|t|_{\mathbb{A}}^{s}\,d^\times t \leftarrow \text{entre in } s,$$

$$+ \cancel{\int_{|t|\le 1} \frac{\hat f(0)}{|t|}\,\omega(t)\,|t|^{s}\,d^\times t}^{\,0} - \cancel{\int_{|t|\le 1} f(0)\,\omega(t)\,|t|^{s}\,d^\times t}^{\,0}.$$

$$\hat{f}(0) \int_{t_\infty \in \mathbb{R}} \left\{ \begin{matrix} 1 \\ \operatorname{sgn} t_\infty \end{matrix} \right\} |t_\infty|^{s-1} \frac{dt_\infty}{|t_\infty|} \cdot \times$$

$$\times \prod_p \int_{t_p \in \mathbb{Z}_p^\times} \underbrace{\omega_p(t_p)} |t_p|_p^{s-1} \underbrace{d^\times t_{p,}}_{} = \frac{dt_p}{c|t_p|}$$

If p unramified $\omega_p = 1$ on $\mathbb{Z}_p^\times$, local ζ factor.

If p ramified: $\underbrace{\sum_{m \in (\mathbb{Z}/p^e\mathbb{Z})^\times} \omega_p(m)}_{\overset{\shortparallel}{0}} \cdot \underbrace{\int_{u \in \mathbb{Z}_p} \omega_p(1+p^e u)\, du}_{=1} \cdot C$

$\overline{\chi}(m)$

$$\mathbb{Z}_p^\times = (\mathbb{Z}/p^e\mathbb{Z})^\times \times (1+p^e\mathbb{Z}_p)$$

$$\underbrace{t_p} = \underset{\substack{\uparrow \\ (\mathbb{Z}/p^e\mathbb{Z})^\times}}{m} \cdot (1+p^e \underset{\substack{\uparrow \\ \mathbb{Z}_p}}{u})$$

$$\underbrace{\mathcal{J}(s)} \cdot L(\chi, s) \overset{\operatorname{Re} s > 1}{=} \int_{\mathbb{A}^\times/\mathbb{Q}^\times} \vartheta_f(t)\,\omega(t)\,|t|^s\, d^\times t$$

$$\overset{\shortparallel}{\hat{\mathcal{J}}(1-s)} \cdot L(\bar{\chi}, 1-s).$$

$$= \int_{\substack{\mathbb{A}^\times \\ |t| \geq 1}} \underbrace{\left[\vartheta_f(t)\,\omega(t)\,|t|^s + \vartheta_{\hat{f}}(t)\,\overline{\omega(t)}\,|t|^{1-s} \right] d^\times t}_{\text{entire.}}$$

~~Analytic~~ Analytic Cont & FE of $L(\chi, s) \to L(\bar{\chi}, 1-s)$.

Tate observed: local L-function also have FEs.

Def: Given $\omega_p : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ local char, & $f_p \in S(\mathbb{Q}_p)$ (i.e. loc const, cpt supp),

let $$L_p(s, \omega_p, f_p) := \int_{\mathbb{Q}_p^\times} f_p(t_p)\, \omega_p(t_p) |t_p|_p^s \, d^\times t_p.$$

This will have analytic cont & FE, indep of choice of f_p. i.e. for $g_p \in S(\mathbb{Q}_p)$,

Thm: $$\frac{L_p(s, \omega_p, f_p)}{L_p(1-s, \overline{\omega}_p, \hat{f}_p)} = \frac{L_p(s, \omega_p, g_p)}{L_p(1-s, \overline{\omega}_p, \hat{g}_p)}.$$

Recall: If ω_p unramified & $f_p = \mathbb{1}_{\mathbb{Z}_p} \ (= \hat{f}_p)$

then $L_p(s, \omega_p, f_p) = \dfrac{1}{1 - \frac{\chi(p)}{p^s}}$.

"p-adic Gaussian"