<u>Last time:</u> Dirichlet Characters

$\chi : (\mathbb{Z}/N\mathbb{Z})^\times \to \mathbb{C}^\times$.

<u>Rep viewpoint:</u> $G = (\mathbb{Z}/N\mathbb{Z})^\times$.

<u>Eg:</u> $N = 12$, $\quad$ " $\{1, 5, 7, 11\}$.

$$V = L^2(G, \mu) = \left\{ f : G \to \mathbb{C} \;\middle|\; \int_G |f|^2 d\mu \;\underset{''}{} \sum_{a \in G} |f(a)|^2 < \infty \right\},$$

"Hilbert space" $\quad V \cong \mathbb{C}^{|G|}$

$$f : \begin{pmatrix} 1 \\ 5 \\ 7 \\ 11 \end{pmatrix} \mapsto \begin{pmatrix} \pi \\ \sqrt{2} \\ i \\ -1 \end{pmatrix} \in L^2(G).$$

$f, f' \in V$, $\quad \langle f, f' \rangle$

$$\sum_{a \in G} f(a) \overline{f'(a)} = \int_G f \overline{f'} \, d\mu$$

G <u>acts</u> on V, i.e. $\forall g \in G$, $\pi(g) \in \mathrm{Aut}(V)$.

$\pi = (G, V)$ is a <u>representation</u>, $\pi : G \to GL(V)$

$$(\pi(\overset{G}{g}).f)(h) = f_g(\underset{G}{h}) = f(hg) \quad \text{"right-regular representation"}$$

$$\pi(5)f = f_5 : \begin{pmatrix}1\\5\\7\\11\end{pmatrix} \mapsto f\begin{pmatrix}5\\1\\11\\7\end{pmatrix} = \begin{pmatrix}\sqrt{2}\\ \pi\\ -1\\ i\end{pmatrix}.$$

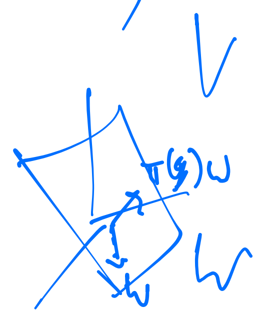
$$\boxed{\begin{aligned} &\pi(gh) = \pi(h)\circ\pi(g) \\ &\pi(gh)f(k) = f(kgh) \end{aligned}}$$

"group basis" for V: $1^{\in G} \leftrightarrow f = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ $g=1$

<u>Want</u>: Find a better basis for action of G.

<u>Def.</u> $W \subset V$ subspace, If G preserves W,

i.e. $\forall g \in G$, $\forall w \in W$, $\pi(g)w \in W$.



then say (G,W) is a subrepresentation of V.
& we say W is G-invariant.

<u>Def.</u> (π, W) is <u>irred</u> if it has no proper G-invariant subspace.

<u>Ex:</u> $V = L^2(G)$, $G = (\mathbb{Z}/N\mathbb{Z})^\times \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, $N = 12$.

$$5: \begin{pmatrix} 1 \\ 5 \\ 7 \\ 11 \end{pmatrix} \mapsto \begin{pmatrix} 5 \\ 1 \\ 11 \\ 7 \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ & & 0 & 1 \\ & & 1 & 0 \end{pmatrix}}_{\pi(5)} \begin{pmatrix} 1 \\ 5 \\ 7 \\ 11 \end{pmatrix}$$

$$7: \begin{pmatrix} 1 \\ 5 \\ 7 \\ 11 \end{pmatrix} \mapsto \begin{pmatrix} 7 \\ 11 \\ 1 \\ 5 \end{pmatrix}$$

$\forall v, w \in V$, $\cancel{\forall g \in G}$,
$\langle v, w \rangle$
$\shortparallel$
$\langle \pi(g)v, \pi(g)w \rangle$

Try $e_1 = 1 + 5 + 7 + 11 = \mathbb{1}_1 + \mathbb{1}_5 + \mathbb{1}_7 + \mathbb{1}_{11} = \text{Const.}$

$W_1 = \mathbb{C}e_1$ is a subspace & G-invariant.

Claim: If $W \subset V$ & G-invariant ($\&\ \pi = \text{unitary}$)

then so is $W^\perp$. i.e. $V = W \oplus W^\perp$.

pf: $W^\perp = \{ v \in V : \forall w \in W,\ \langle v, w \rangle = 0 \}$.

Take $\forall v$. Take any $g \in G$. Claim: $\pi(g).v \in W^\perp$.

$\Leftrightarrow \forall w \in W, \quad \langle \pi(g) v, w \rangle \overset{?}{=} 0$.

π unitary action $\curvearrowright$ $= \langle \cancel{\pi(g^{-1})\pi(g)} v, \pi(g^{-1}) w \rangle$

$= \langle v, \underbrace{\pi(g^{-1}) w}_{\in W} \rangle = 0$ $v \in W^\perp$.

$\Leftarrow$ W is G-inv.

$\Rightarrow \pi(g) v \in W^\perp$

Main Goal of rep theory: decompose

$G \curvearrowright L^2(G)$ into irreducibles (& stitch reps back together).

__E.g.:__ $G = S^1 = \mathbb{R}/\mathbb{Z} \subset \mathbb{C}$, $L^2(G, \overset{\text{lebesgue}}{\mu}) = \bigoplus_{n \in \mathbb{Z}} \underbrace{\mathbb{C} \cdot e_n}_{\text{irreducible subspaces}}$.

$e_n(x) = e^{2\pi i n x}$

$f \in$

$= \sum_{n \in \mathbb{Z}} \underbrace{\langle f, e_n \rangle}_{\hat{f}(n)} \cdot e_n$.

__E.g.:__ $G = \mathbb{R}$, $L^2(G, \mu) = L^2(\mathbb{R}, dx)$.

$\underline{e_\xi(x) = e^{2\pi i x \xi}}$.

$= \int_{\xi \in \mathbb{R}} \mathbb{C} e_\xi$

__Not__ in $L^2(G)$. "Continuous spectrum".

Eg: $\mathbb{A}^\times \curvearrowright L^2(\mathbb{Q}^\times \backslash \mathbb{A}^\times, \mu)$ decomposition into irred.

Eg: $GL(n,\mathbb{A}) \curvearrowright L^2\left(\dfrac{GL(n,\mathbb{A})}{GL(n,\mathbb{Q})}, \mu\right)$

Eg: $G(\mathbb{A}) \curvearrowright L^2\left(G(\mathbb{A})/G(\mathbb{Q})\right)$

Back to $L^2((\mathbb{Z}/12\mathbb{Z})^\times) \ni e_1 = 1+5+7+11$ ✓

$W_1 \oplus W_2 \oplus W_3 \oplus W_4$.

$1: e_2 \mapsto e_2$

$5: e_2 \mapsto e_2$.

$e_2 = 1+5-7-11$ ✓

$e_3 = 1-5+7-11$

$e_4 = 1-5-7+11$.

$7: e_2 \mapsto (7+11-1-5) = -e_2$.

Def: If $\pi: (G, W)$ is a G-rep,

$\chi = \chi_\pi : G \to \mathbb{C} : g \mapsto tr(\pi(g))$

What is $\chi(G, W_2)$ above?

$$\chi\begin{pmatrix}1\\5\\7\\11\end{pmatrix} = \begin{pmatrix}1\\1\\-1\\-1\end{pmatrix},$$

Def. If $\pi_1 = (G, V_1)$, & $\pi_2 = (G, V_2)$,

if $\iota : V_1 \to V_2$ homomorphism s.t.

$$\iota \circ \pi_1 = \pi_2 \circ \iota$$

$$\begin{tikzcd} v_1 \in V_1 \arrow[r, "\pi_1(g)"] \arrow[d, "\iota"'] & \pi(g)\, v_1 \in V_1 \arrow[d, "\iota"] \\ \iota v_1 \in V_2 \arrow[r, "\pi_2(g)"] & \iota(\pi(g)\, v_1) \in V_2 \end{tikzcd}$$

$$\iota(\pi(g)\, v_1) = \pi_2(g)\, \iota v_1$$

then ι is an "intertwining operator"

Def. If ι is isomorphism (bij) then we say: $\pi_1 \cong \pi_2$ isomorphic.

$\hat{G}$ = unitary dual = {unitary irreducible reps}/$\sim$.

Lemma (Schur): If $\pi_1(G, W_1)$ & $\pi_2(G, W_2)$ is irred.

& $\iota: W_1 \to W_2$ homom. intertwines π_1 & π_2

& $\iota \neq 0$,

then ι is isomorphism.

pf: If ι not surjective, $\mathrm{Im}(\iota) < W_2$

is G-invar. But W_2 = irred.

If ι not inj, $\mathrm{Ker}(\iota) \subset W_1$

is G-invar but W_1 = irred.

Lemma (Schur): If $\pi = (G, W)$ irred & fin dim'l

$\iota: W \to W$ intertwining & non-zero, then

$\exists c \in \mathbb{C}: \iota = c \cdot I$

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pf: ι has at least one eigenvalue $c \in \mathbb{C}$,

$\iota - c$ has Ker dim ≥ 1. & Ker = G-inv.

$\Rightarrow \ker = W$.

<u>Lemma:</u> If $\underline{G = \text{abelian}}$ & $\pi = (G, W)$ irred, then $\dim W = 1$.

<u>pf:</u> Let $g, h \in G$, $L = \underline{\pi(g)} : W \to W$.

$$\pi(h)\pi(g) = \pi(hg) = \pi(gh) = \pi(g)\pi(h),$$

$\pi(h) \circ L = L \circ \pi(h)$ so $\pi(g)$ intertwines π with itself.

$\Rightarrow \pi(g) = c_g \cdot I$.

any $v \in W$ spans an invariant subspace.

$\Rightarrow \mathbb{C}v$ "

<u>Eg:</u> Non-abelian $G = SL_2(\mathbb{F}_2)$.

$$G = \left\{ \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{\text{ord } 2}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}^{\text{ord } 2}, \right.$$

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$$\begin{pmatrix}0&1\\1&0\end{pmatrix}, \begin{pmatrix}1&1\\1&0\end{pmatrix}, \begin{pmatrix}0&1\\1&1\end{pmatrix} \Big\} \simeq S_3,$$

ord 2   ord 3

$$S_3 = \langle a, b : a^3 = 1 = b^2,\ ba = a^2 b \rangle = G$$

$$L^2(G) = \mathbb{C},$$

group basis: e, a, a^2, b, ab, a^2b

Find "nice" basis for $L^2(G)$ giving invariant G-spaces. $L^2(G) = W_1 \oplus W_1^{\perp}$.

$$e_1 = e + a + a^2 + b + ab + a^2 b.$$

$$e_2 = e - a - a^2 + b + ab + a^2 b.$$

(G, V) reg rep, $V \cong W_1 \oplus \cdots \oplus W_k$ irred, (not nec non-iso).

<u>Claim</u>: # times $\cong W_k$ occur in $\oplus$ = $\dim W_k$.

Fact: $\chi_V \begin{pmatrix} e \\ a \\ a^2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \dim V = |G| \\ 0 \\ 0 \\ 0 \end{pmatrix}$,

$\sum_j \chi_{W_j}$

Fact: If W_1 not isom to W_2 irred,

then $\langle \chi_{W_1}, \chi_{W_2} \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_{W_1}(g) \overline{\chi_{W_2}(g)}$.

$= \begin{cases} 0 & W_1 \neq W_2 \\ 1 & \text{else.} \end{cases}$

So $\langle \chi_{W_j}, \chi_V \rangle = \sum_i \langle \chi_{W_j}, \chi_{W_i} \rangle$,

$|G|$

$= \#$ times W_j occurs in V

$\frac{1}{|G|} |G||G| = \langle \chi_V, \chi_V \rangle = \sum_j \langle \chi_{W_j}, \chi_V \rangle$,

$$= \sum_j \#\text{times}$$

Next time: If $W_1, \ldots, W_j$ are distinct irred occurring in $V = \text{reg}$,

then $\sum (\dim W_i)^2 = |G|$.