Review: $\sum_{p\equiv a(q)} \frac{1}{p} = \infty$, $\Leftarrow L(1,\chi_q) \neq 0$ $\Leftarrow L(1,\chi) = \frac{\pi}{\sqrt{q}} \cdot h(q)$ $(q \equiv 3(4))$

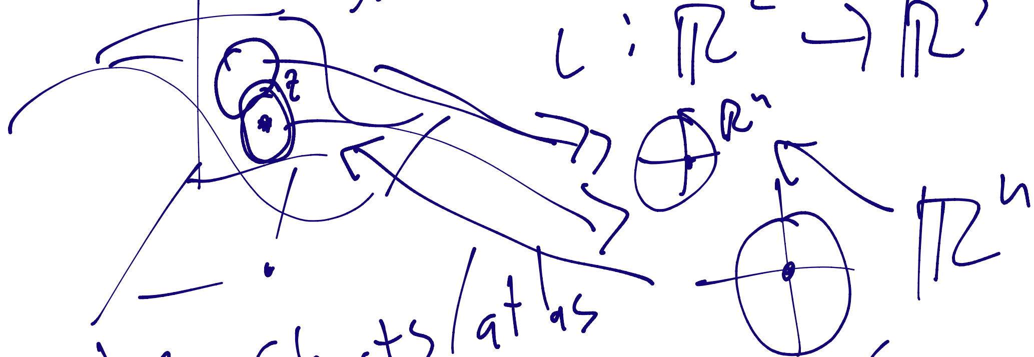
$\chi_q \neq 1$

Saw $h(q) < \infty$, fund domain $\Gamma \circlearrowright \mathbb{H}$, $\Gamma = PSL_2(\mathbb{Z})$ — more structure.

"manifold" 19th Century

Riemann manifold

$\iota : \mathbb{R}^2 \to \mathbb{R}^3$



X charts/atlas

(Nash embedding).

$\gamma : p \to q$.

$\gamma(0) = p$ $q = \gamma(1)$

$\gamma : [0,1] \to X$

$L(\gamma) = \int_0^1 \underbrace{|\gamma'(t)|}_{\sqrt{Q(\gamma'(t))}} dt$

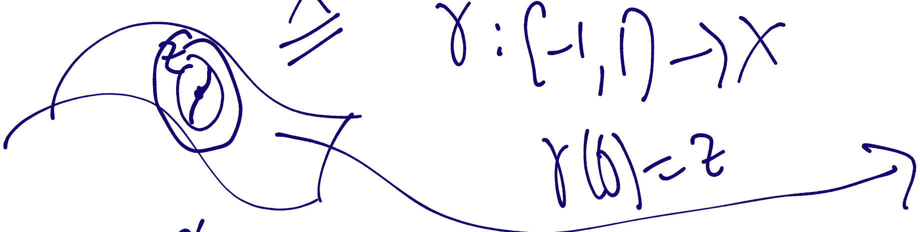
quad form.

Riemannian Structure: "locally" measure lengths via Q

X

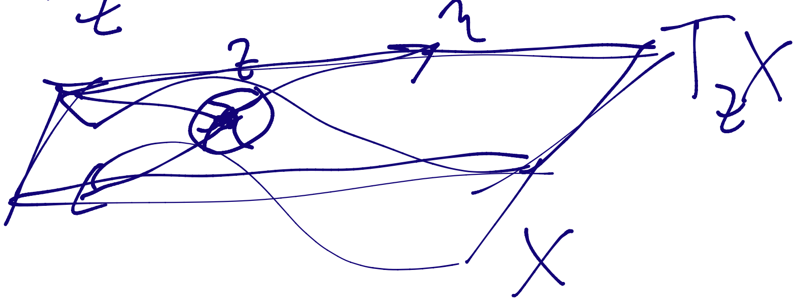
$\gamma : [-1,1] \to X$

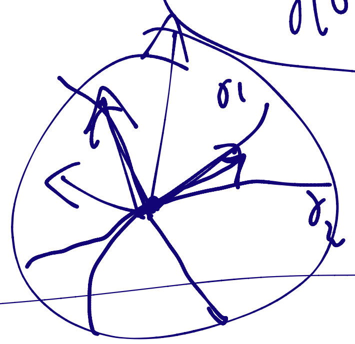
$\gamma(0) = z$



Idea: C^1-curves, γ_1, γ_2 thru z are $\sim$ $\Leftrightarrow$ $\gamma_1'(0) = \gamma_2'(0)$

T_zX = tangent space to X at z = $\left\{ \gamma : (-1,1) \to X, \ \gamma(0) = z \right\} / \sim$.

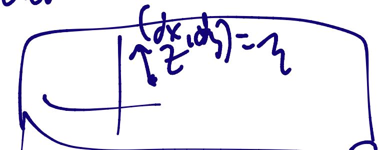




$TX = \bigsqcup_{z \in X} T_zX$.

Riemann structure

at each T_zX, q_z quad form on tangent vectors.

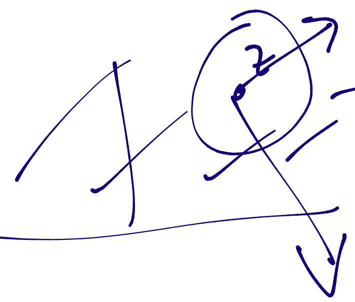


$L(\gamma) = \int \sqrt{q_z(\gamma'(t))}\, dt$.

Eg $X = \mathbb{R}^2$. $T_zX = \mathbb{R}^2$

$\forall z$, $q_z(d_x, d_y) = d_x^2 + d_y^2 = d_z^2$, $\xi = (d_x, d_y)$

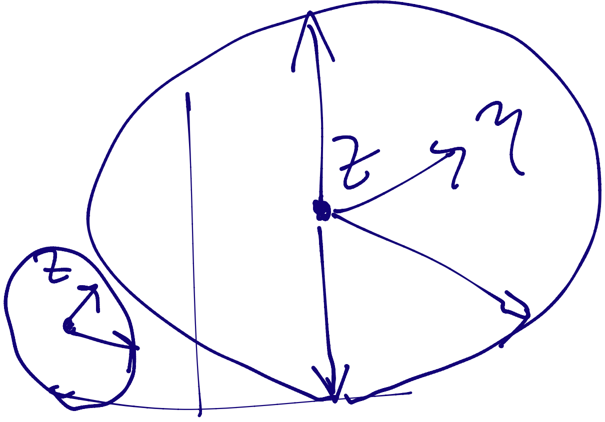
For $\mathbb{H}$.



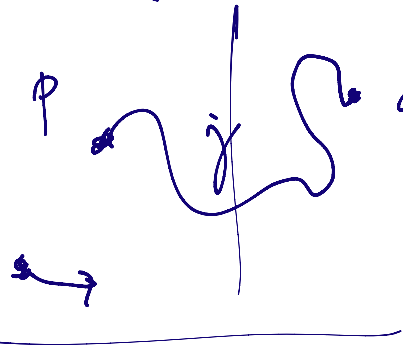
charts: identify to $\mathbb{R}^2 = \mathbb{C}$

$T_z\mathbb{H} = \mathbb{C} \ni \xi$, hyperbolic structure: $q_z(\xi) = \frac{|\xi|^2}{\operatorname{Im} z^2}$

$T^1\mathbb{H}$ = unit tangent bundle = $\left\{ (z, \xi) \in T\mathbb{H}, \ \|\xi\|_z = 1 \right\}$

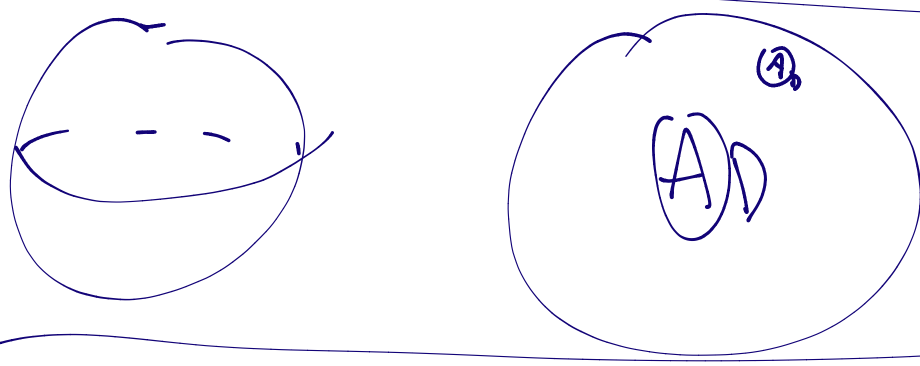


$$\frac{|\eta|}{y} = 1, \quad |\eta| = y.$$



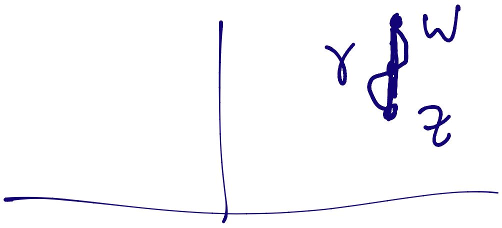
Geodesics: Look at all curves $\gamma : p \Rightarrow q$, distance $\underbrace{d(p,q) = \inf_{\gamma} L(\gamma)}$.

a geodesic $p \Rightarrow q$ is a curve with $L(\gamma) = d(p,q)$.



Who are geodesics in $\mathbb{H}$?. $\gamma = (x(t), y(t))$.

Easy case:



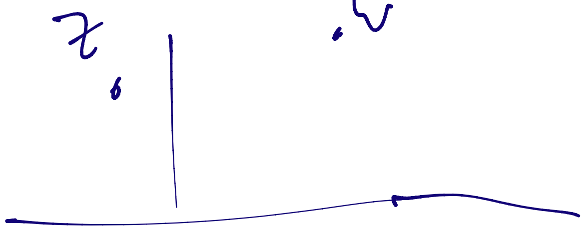
$$L(\gamma) = \int_0^1 \sqrt{\frac{\dot{x}(t)^2 + \dot{y}(t)^2}{y(t)^2}}\, dt$$

$$\geq \int_0^1 \frac{|\dot{y}(t)|}{y(t)}\, dt.$$

$$\geq \int_0^1 \frac{\dot{y}(t)}{y(t)}\, dt$$

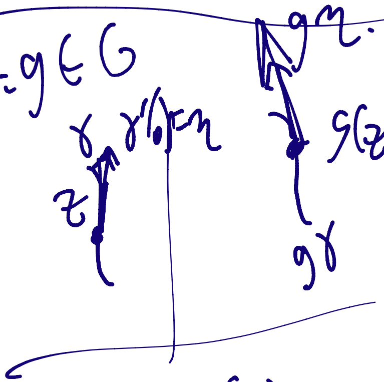
$\Rightarrow$ vertical lines are geodesics.
If $\operatorname{Im} z = \operatorname{Im} w$.

Action of $G = (P)SL_2(\mathbb{R})$ on $\mathbb{H}$



extends to action on $\underline{T^1\mathbb{H}}$.

$\begin{pmatrix}a&b\\c&d\end{pmatrix} = g \in G$



$D_z g$

$(g\gamma)'(0) = g'(\underbrace{\gamma(0)}_{z}) \cdot \underbrace{\gamma'(0)}_{\eta}$

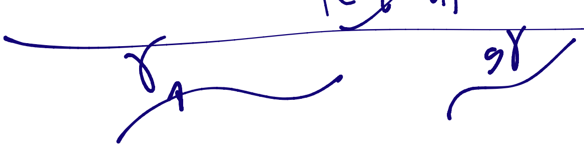
$$g(z) = \frac{az+b}{cz+d}, \quad g'(z) = \frac{(cz+d)a - (az+b)c}{(cz+d)^2} = \frac{1}{(cz+d)^2}$$

$$g(z,\eta) = \left(\frac{az+b}{cz+d}, \frac{\eta}{(cz+d)^2} \right), \quad G \overset{?}{\curvearrowright} T^1\mathbb{H},$$

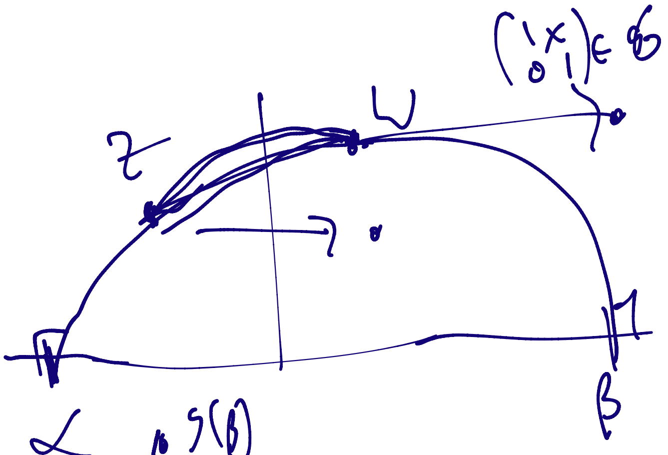
Is this an isometry? Assume $(z,\eta) \in T^1_z\mathbb{H}$.

$$1 = \|\eta\|_z = \frac{|\eta|}{\operatorname{Im} z} \overset{?}{=} \|g\eta\|_{gz} = \frac{\left|\frac{\eta}{(cz+d)^2}\right|}{\operatorname{Im} gz}$$

$$= \frac{|\eta|}{\cancel{|cz+d|^2}} \cdot \frac{\cancel{|cz+d|^2}}{\operatorname{Im} z}.$$

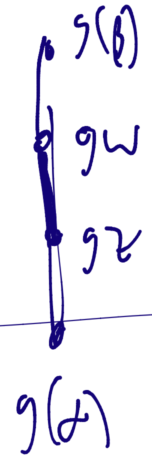


$$L(g\gamma) = L(\gamma) \qquad \int \underbrace{\|(g\gamma)'\|_{g\gamma}}_{= \|\gamma'\|_\gamma} dt.$$

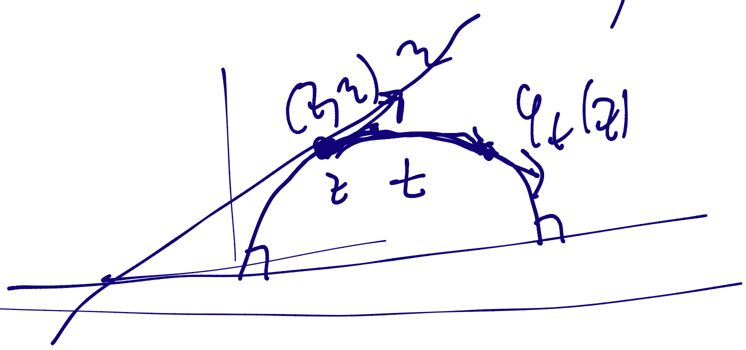


Apollonius: $\exists!$

$\exists\ g\in G$ s.t. $g(\alpha)=0,\ g(\beta)=\infty.$

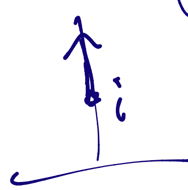


Given $(z,\xi)\in T^1\mathbb{H}$, define geodesic flow φ_t,



Gelfand reinterprets this action algebraically:
$PSL_2(\mathbb{R})$.

Identify $G \cong T^1\mathbb{H}$. $G/K=\mathbb{H}$,



transitive
$\mathrm{Stab}_G(i,\uparrow)=e.$

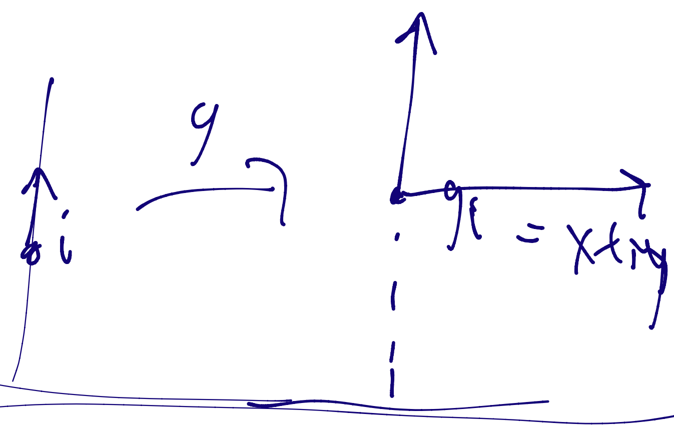
$$g\mapsto g(i,\uparrow)=\left(\frac{ai+b}{ci+d},\ \frac{i}{(ci+d)^2}\right).$$

Let $g=\underbrace{\begin{pmatrix}1&x\\0&1\end{pmatrix}}_{n_x}\underbrace{\begin{pmatrix}\sqrt{y}&0\\0&1/\sqrt{y}\end{pmatrix}}_{a_y}\underbrace{\begin{pmatrix}\cos\theta&\sin\theta\\-\sin\theta&\cos\theta\end{pmatrix}}_{k_\theta}.$

Then

$$g(i, i) = \left(\underbrace{\underbrace{n_x a_y \cancel{k_\theta}\, i}_{y_i}}_{x+iy}, \; \underbrace{\frac{i}{\left(\frac{-\sin\theta}{\sqrt{y}}\, i + \frac{\cos\theta}{\sqrt{y}}\right)^2}}_{\frac{y_i e^{i2\theta}}{\cancel{(\cos\theta - i\sin\theta)}}} \right)$$

$$g = \begin{pmatrix} \sqrt{y} & \cancel{x} \\ 0 & 1/\sqrt{y} \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} & \\ \underbrace{-\frac{\sin\theta}{\sqrt{y}}} & \underbrace{\frac{\cos\theta}{\sqrt{y}}} \end{pmatrix}$$



1- param subgroups

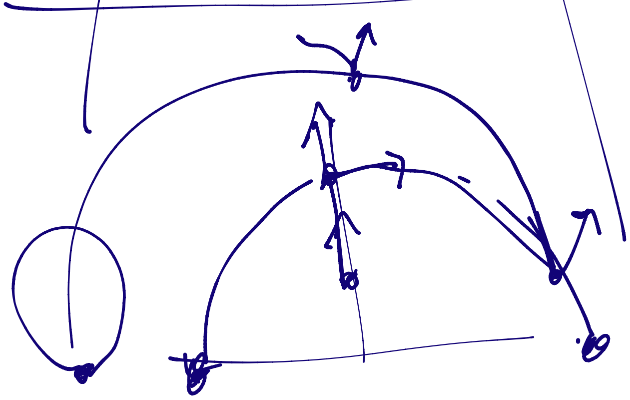
$3^3 = 27$ decomps.

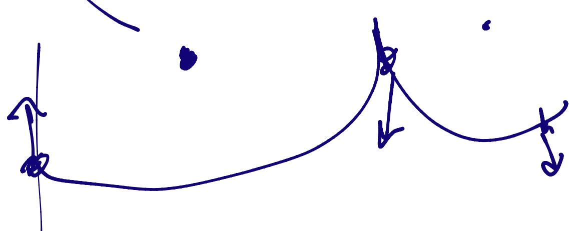
$gz = z$.

$\frac{az+b}{cz+d} = z$

$\simeq N$, $\simeq A$, $\simeq K$.

$x \begin{pmatrix} 1 & \cancel{x} \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} e^{i\theta} & \\ & e^{-i\theta} \end{pmatrix}$.





HAK

Under this identification, Gelfand realized

$\varphi_t \circlearrowright T^1\mathbb{H} \Longleftrightarrow$ right-regular action of a_t

$$a_t = \begin{pmatrix} e^{t/2} & \\ & e^{-t/2} \end{pmatrix} = \exp\left(t\begin{pmatrix} 1 & \\ & -1 \end{pmatrix}\right)$$

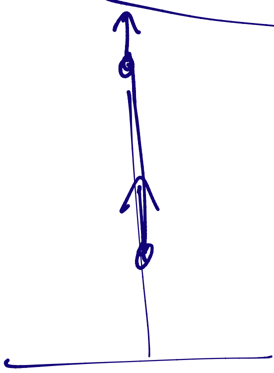
on G.

$\underbrace{\pi(h)\cdot g}\, i$

$\gamma \cdot g h \cdot (i, \uparrow)$

a_t

nak



$$\sinh x = \frac{e^x - e^{-x}}{2}$$

Exercise: $\boxed{\sinh\left(\frac{d_{\mathbb{H}}(z,w)}{2}\right) = \frac{u(z,w)}{2}}$, where

$$u(z,w) = \frac{|z-w|}{2\sqrt{\operatorname{Im} z \operatorname{Im} w}}$$

Hint: $u(gz, gw) = u(z,w)$.