Recall: $L_\chi(s) = \sum_{n\geq 1} \frac{\chi(n)}{n^s}$. anal. cont, FE $s \mapsto 1-s$ $(\chi \to \bar{\chi})$.

If χ primitive

$$\chi(n) = \frac{1}{\sqrt{q}} \sum_{m(q)}{}' \widehat{\underbrace{\chi(m)}_{\chi(\bar{m})\,\hat{\chi}(1)}}\, e_q(-nm)$$

$(|\tau(\chi)| = \sqrt{q})$

$\hat{\chi}(1) = \frac{\tau(\chi)}{\sqrt{q}}$.

$$= \frac{\tau(\chi)}{q} \sum_{m(q)}{}' \chi(\bar{m})\, e_q(-nm)$$

$$\boxed{\chi(n)} = \underbrace{\frac{\chi(-1)\,\tau(\chi)}{q} \sum_{m(q)}{}' \chi(\bar{m})\, e_q(nm)}$$

How to discover Dirichlet class number formula?

$$L(1,\chi) = \sum_{n\geq 1} \frac{\boxed{\chi(n)}}{n}$$

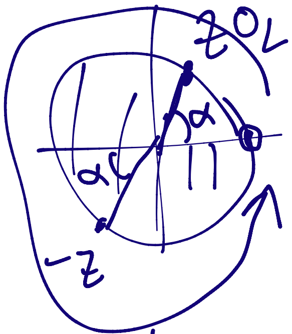
Conditionally converges, $F(x) = \sum_{n<x} \chi(n) \ll_q 1$.

$$\text{"="} \frac{\chi(-1)\,\tau(\chi)}{q} \underbrace{\sum_{m(q)}{}'}_{0<m<q.} \chi(\bar{m}) \underbrace{\sum_{n\geq 1} \frac{e_q(nm)}{n}}_{-\log(1-z)}.$$

$z = e^{2\pi i m/q}$

$$z+\frac{z^2}{2}+\frac{z^3}{3}+\dots = -\log_{\mathbb{C}}(1-z), \qquad w=|w|e^{i\theta}$$

$0<\alpha<2\pi$.



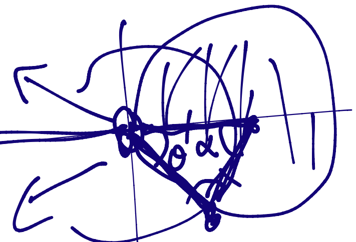
$|z|\le 1$

$z\ne 1$

$\boxed{\alpha=\frac{2\pi m}{q}.}$

$w=1-z$

$\theta=\arg w.$



$w=1-z$

$\underbrace{|\theta|<\pi}$

$$\log_{\mathbb{C}} w = \log_{\mathbb{R}}|w| + i\theta.$$

$$c^2=a^2+b^2-2ab\cos C.$$

$$|w|^2 = 1^2+1^2-2\cdot 1\cdot 1\cdot \cos\alpha = 2-2\cos\alpha$$

$$\cos 2\eta = \cos^2\eta - \sin^2\eta = 1-2\sin^2\eta.$$

$$= 2-2\left(1-2\sin^2\frac{\alpha}{2}\right) = 4\sin^2\frac{\alpha}{2}.$$

$$\boxed{|w| = 2\left|\sin\frac{\alpha}{2}\right|}$$

Isosceles $\Rightarrow$

$$\theta+\theta+\alpha=\pi \Rightarrow \theta=\frac{\pi-\alpha}{2}.$$

$$\pi-\alpha=\frac{\pi}{q}(q-2m).$$

$$\boxed{\alpha=\frac{2\pi m}{q}.}$$

$$\log_{\mathbb{C}} w = \log\left|2\sin\frac{\alpha}{2}\right| + i\left(\frac{\pi-\alpha}{2}\right).$$

$$L(1,\chi) = -\frac{\chi(-1)\tau(\chi)}{q}\sum_{0<m<q}\bar{\chi}(m)\left[\log\left|2\sin\frac{\pi m}{q}\right| + \frac{i\pi}{2q}(q-2m)\right]$$

$\chi(-1)=-1$ if $q\equiv 3$

Exercise: If $\chi=\left(\frac{\cdot}{q}\right)$, $q\equiv 3\ (4)$ prime $\Rightarrow \tau(\chi)=i\sqrt{q}$.

E.g.: $q=7$,

n	1	2	3	4	5	6
χ	1	1	-1	1	-1	-1

Exercise $\leftarrow$ $i\sqrt{7}$.

$$t(\chi) = e_7(1) + e_7(2) - e_7(3) + e_7(4) - e_7(5) - e_7(6).$$

If $q \equiv 3 (4)$,

$$L(1,\chi) = -\frac{(-1)\, i\sqrt{q}}{q} \sum_{0<m<q} \chi(m) \left[\cancel{0q} \cdots - \frac{2i\pi m}{2q}\right].$$

$$\mathrm{Re}(L(1,\chi)) = -\frac{\pi}{\sqrt{q}} \cdot \boxed{\frac{1}{q} \sum_{0<m<q} \chi(m)\cdot m}.$$

Can this sum ever vanish? (For primes in prog.)

Eg: $q=7$: $L(1,\chi_7) = -\frac{\pi}{\sqrt{7}}\, \frac{1}{7}\left[\underbrace{1+2-3}+\underbrace{4-5-6}\right]$.

$$\frac{1}{1}+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}-\frac{1}{6}+\frac{0}{7}\cdots\cdots = \frac{\pi}{\sqrt{7}}.$$

-7

My guess: Dirichlet computed ton of these, & noticed pattern with Gauss class numbers.

Exercise: Compute $L(1,\chi_{23}) = \frac{\pi}{\sqrt{23}} \cdot 3$.

$h(-23) = 3$.

Need: Gauss (Legendre) Binary Quadratic Forms.

Idea: Fermat 1637(?): What numbers are rep'd by binary quadratic form $x^2+y^2 = Q(x,y)$.

(Read Cox: Primes of the form x^2+ny^2).

General: Which n's rep'd by $Q = [A,B,C] = Ax^2+Bxy+Cy^2$.

Obs: If $Q'(x,y) := Q(ax+by, cx+dy)$

is an invertible ($\mathbb{Z}$) linear transf: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$ Then Q' represents same numbers as Q. $\in GL_2(\mathbb{Z})$.

Def: $Q' \sim Q$ are properly equivalent if $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$.

Exercise: Let $D_Q := B^2-4AC$. Then $Q \sim Q' \Rightarrow D_Q = D_{Q'}$.

Is converse true? If $D_Q = D_{Q'} \overset{?}{\Rightarrow} Q \sim Q'$?

Let $[Q] = \{Q' \sim Q\}$ be the class of Q.

Fix D. Let $\mathcal{C}_D = \{ [Q] \mid D_Q = D \}$ ← class "group".

Thm (Legendre–Gauss): $h(D) := \left| \mathcal{C}_D \right| < \infty$.

class number.

Note: If $D = B^2 - 4AC$. $\Rightarrow D \equiv D\ (4)$, $\Rightarrow D \equiv 0$ or $1\ (4)$

Big difference between $D>0$, $D<0$.

(or $D = \square$, $D=0 \Rightarrow$ Q factors$/\mathbb{Q}$ $Q = (x+y)(x-y)$.)

If $D>0$, $Q(x,y) = Ax^2 + Bxy + Cy^2$

Let $\alpha_Q = \dfrac{-B+\sqrt{D}}{2A}$ $\quad = A(x - \alpha_Q y)(x - \overline{\alpha}_Q y)$.

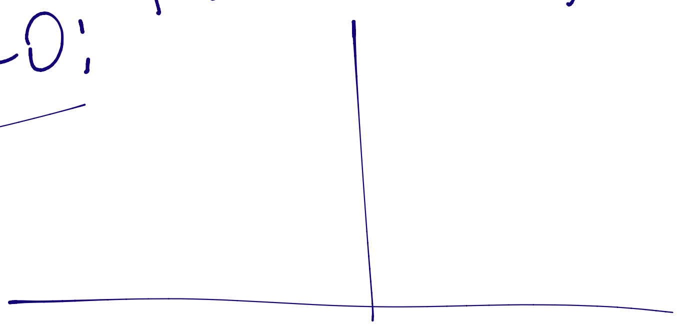
$Q(\alpha_Q, 1) = 0$. Look over $\mathbb{R}$ $Z = Q(x,y)$.

$\alpha_Q \in \mathbb{R}$ $D>0$ • $Q(x,y) = x^2 - y^2$., $B^2 - 4AC = 4 > 0$. takes both pos & neg values. = "indefinite".

$\alpha_Q \in \mathbb{H}$ • $D<0$ $Q(x,y) = x^2 + y^2$ takes only one sign = "pos/neg. definite"

Proof that $h(D) < \infty$ by reduction theory:

If $D<0$:



• α_Q. Take any Q,

If $Q \mapsto Q' = Q \circ \gamma$, $\gamma \in SL_2(\mathbb{Z})$, $\alpha_{Q'} = \underbrace{\gamma^{-1} \circ \alpha_Q}$ (check?)

$$\boxed{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \circ \alpha = \frac{a\alpha + b}{c\alpha + d}.}$$

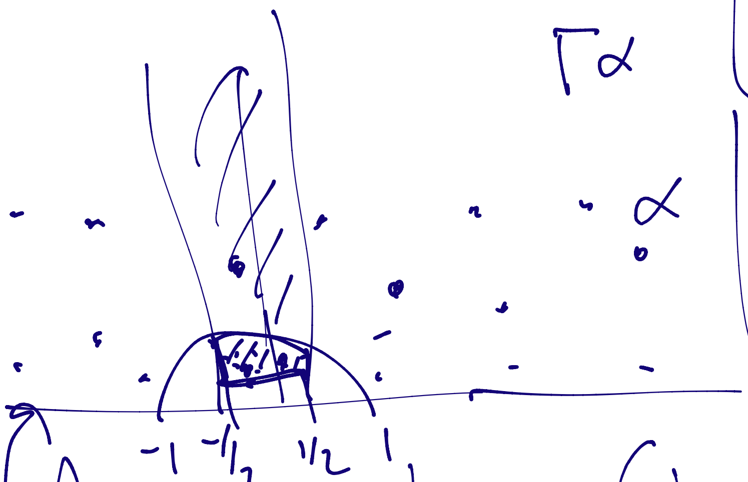
Fund domain for action of $SL_2(\mathbb{Z}) = \Gamma$ on $\mathbb{H}$.

fractional linear action

Aside: $\mathbb{P}^1 = (\alpha, \beta)/\sim$ $= \{(\alpha, 1)\} \cup (1, 0)$. $(\alpha,\beta) \sim (\lambda\alpha, \lambda\beta)$

linear action on $\mathbb{P}^1$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha \\ 1 \end{pmatrix} = \begin{pmatrix} a\alpha + b \\ c\alpha + d \end{pmatrix} \sim \begin{pmatrix} \frac{a\alpha+b}{c\alpha+d} \\ 1 \end{pmatrix}$$



Action is discrete (discontinuous: $\forall K_1, K_2 \subset \mathbb{H}$ cpt, $|\{\gamma \in \Gamma : \gamma K_1 \cap K_2 \neq \emptyset\}| < \infty$.)

max $\mathrm{Im}(\gamma\alpha)$ ← among these, minimize Re.

α, $\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \alpha = \alpha + n$. To make $|\mathrm{Re}| < \frac{1}{2}$,

If $|\;| < 1$, apply $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : z \mapsto \frac{-1}{z}$,

increases Im.

$$\boxed{\mathcal{F} = \{ z \mid |z| > 1, |\mathrm{Re}(z)| < \tfrac{1}{2} \}}$$

Exercise: If $z, z' \in \mathcal{F}$ & $\gamma z = z' \Rightarrow$
$z = z'$ & $\underline{\gamma = I}$ in PSL_2

$D < 0$ | Assume $A > 0$.

Why is $h(D) < \infty$ | Any $[Q]$ has rep Q with $\alpha_Q \in \mathcal{F}$.

$$= \frac{-B + \sqrt{D}}{2A}$$

$$\frac{|\sqrt{D}|}{3A} \geq \left| \frac{-B + i\sqrt{|D|}}{2A} \right| > 1 \quad \& \quad \left| \frac{-B}{2A} \right| < \frac{1}{2}, \quad \underbrace{|B| < A}$$

$$\frac{B^2 - (B^2 - 4AC)}{4A^2} > 1 \Rightarrow \frac{C}{A} > 1$$

$\underbrace{0 < A < \sqrt{|D|}}$ Given D, finitely many A's.
finitely many B's.

$\boxed{\text{Fix next time}}$ C det'd by $D = B^2 - 4AC$.

Eg: $D = -23 \equiv 1 \ (4)$. $\sqrt{|D|} < 5$

$A = 1$, $B = -1, \cancel{0}, 1$, $\quad -23 = 1 - 4 \cdot 1 \cdot C$, $\quad C = \frac{B^2 - D}{4A}$
$C = 6$.

$A = 2$

$A = 3$

$A = 4$, . . .

$$\boxed{h(-23) = 3}$$

Owe you $D > 0$.