Last time:

$(a,q)=1.$

$$\ell_\chi(s) = \sum_p \sum_k \frac{1}{k p^{ks}}, \quad \exp \ell_\chi(s) = L_\chi(s)$$

$$\frac{1}{\varphi(q)} \sum_{\chi(q)} \overline{\chi(a)}\, \ell_\chi(s) = \sum_{p \equiv a(q)} \frac{1}{p^s} + O(1) \text{ as } s \to 1^+.$$

one $\chi = \chi_0$, $\overline{\chi_0}(a) = 1$

✓ • Need: $L_\chi(s)$ entire $(\Rightarrow \mathrm{Re}\, \ell_\chi(s) \not\to \infty)$

• Need: $\chi=\bar\chi$, $L_\chi(1) \neq 0$ $(\Rightarrow \mathrm{Re}\, \ell_\chi(s) \not\to -\infty)$.

$\ell_{\chi_0}(s) \to \infty$

$\chi \neq \bar\chi$,

$\mathrm{Re}\, \ell_\chi(s) \not\to -\infty$.

Eg: $e^{-\pi t^2}$ $\cdot t e^{-\pi t^2}$

If χ even $(\chi(-1)=1)$, f even, else f odd

Use Riemann's technique, $\Theta_{f,\chi}(t) := \sum_{n \geq 1} \chi(n) f(nt)$.

$(\chi(0)=0)$

Poisson summation: $2\Theta_{f,\chi}(t) = \sum_{n \in \mathbb{Z}} \chi(n) f(nt) = ?$

Need to extend χ from $\mathbb{N} \to \mathbb{C}$ to $\mathbb{R} \to \mathbb{C}$.

Idea: (finite) Fourier analysis: $f : \mathbb{Z}/q \to \mathbb{C}$. $\in L^2(\mathbb{Z}/q) = \mathcal{H}$.

$$\hat f(m) := \frac{1}{\sqrt{q}} \sum_{r(q)} f(r)\, e_q(rm) \leftarrow \text{FT is isometry}$$

Rationale: Claim: Parseval identity: $\|f\|_2^2 = \|\hat f\|_2^2$.

Pf: $\sum_{r(q)} |f(r)|^2$ $\sum_{m(q)} \left| \frac{1}{\sqrt{q}} \sum_{r(q)} f(r)\, e_q(rm) \right|^2$.

open square

$$\sum_{r_1(q)} \sum_{r_2(q)} f(r_1) \overline{f(r_2)} \underbrace{\frac{1}{q} \sum_{m(q)} \overbrace{e_q(r_1 m)\, e_q(-r_2 m)}^{e_q(m(r_1-r_2))}}_{\mathbb{1}_{r_1 \equiv r_2}}.$$

Fourier inversion:

Claim: $f(r) = \frac{1}{\sqrt{q}} \sum_{m(q)} \hat f(m)\, e_q(-mr)$

Exercise: ↑.

Idea: Expand χ as F series. $(\mathbb{Z}/q)$. $(\mathbb{Z}/q)^\times$,

$$\hat{\chi}(m) = \frac{1}{\sqrt{q}} \sum_{r(q)} \chi(r)\, e_q(rm) = \frac{1}{\sqrt{q}} \sum_{r(q)}{}' \chi(r)\, e_q(rm)$$

If $(m,q)=1$; $r \mapsto r\bar{m}$

$$= \frac{1}{\sqrt{q}} \sum_{r(q)}{}' \chi(r\bar{m})\, e_q(r)$$

$$= \boxed{\chi(\bar{m})\, \underline{\hat{\chi}(1)}. = \hat{\chi}(m)}$$

$$I(\chi) = \sum_{r(q)} \chi(r)\, e_q(r) = \sqrt{q} \cdot \hat{\chi}(1).$$

= Gauss sum

$e^{-\pi x^2}$ Gauss distr

Obs: if $\chi(r) = \left(\frac{r}{q}\right) = \begin{cases} 0 & (r,q)>1 \\ 1 & r \equiv \square \\ -1 & \text{else} \end{cases}$, then:

$$I(\chi) = \sum_{r(q)} \underbrace{(\chi(r)+1)\, e_q(r)}_{=0,} = \sum_{a(q)} e_q(a^2)$$

q = prime

$$\boxed{1+\chi = \begin{cases} 1 & r \equiv 0 \\ 2 & r \equiv \square \\ 0 & r \not\equiv \square \end{cases}}$$

$= \#$ sol'ns to $x^2 - r \equiv 0$

If $(m,q)>1$, $\hat{\chi}(m) = \frac{1}{\sqrt{q}} \sum_{r(q)} \chi(r)\, e_q(rm)$.

& χ is primitive $\Rightarrow$ $\hat{\chi}(m) = 0$. (Exercise)

If $q_1 | q$, $\chi_1 \pmod{q_1}$ $\rightarrow$ induce $\chi(q)$ $\chi(n) = \begin{cases} \chi_1(n) & \text{else.} \\ 0 & \text{if } (n,q)>1 \end{cases}$

Exercise: Compute $\hat{\chi}(m)$ for $\chi_2 \pmod{12}$. $m \in \mathbb{Z}/12$.

Eg: $q=12$, $(\mathbb{Z}/12)^\times = \{1,5,7,11\}. = \mathbb{Z}/2 \times \mathbb{Z}/2.$

	1	5	7	11	
χ_0	1	1	1	1	← trivial ⇒ imprimitive.
χ_1	1	1	-1	-1	← primitive
χ_2	1	-1	1	-1	← imprimitive $\chi \pmod 6$,
χ_3	1	-1	-1	1	← primitive.

$$\boxed{\begin{array}{cc} G & > \Gamma \\ \uparrow & \uparrow . \\ G/\Gamma & \end{array}}$$

Fourier inversion:

If χ = primitive $\quad \chi(r) = \frac{1}{\sqrt{q}} \sum_{m(q)}' \underbrace{\hat{\chi}(m)}_{\chi(\bar{m})\frac{\tau(\chi)}{\sqrt{q}}} e_q(-mr)$

Realized mult char $\uparrow$ $= \frac{\tau(\chi)}{q} \underbrace{\sum_{m(q)} \chi(\bar{m})\, e_q(-mr)}$;

as linear comb of additive char's $\quad r \in \mathbb{R}$.

$$2\theta_{\chi_f}(t) = \sum_{n\in\mathbb{Z}} \chi(n) f(nt)$$

$$= \frac{\tau(\chi)}{q} \sum_{m(q)} \chi(\bar{m}) \underbrace{\sum_{n\in\mathbb{Z}} \underbrace{e_q(-mn) f(nt)}}.$$

Poisson

$$= \frac{\tau(\chi)}{q}\sum_{m(q)}\chi(\overline{m})\sum_{k\in\mathbb{Z}}\underbrace{\int_{\mathbb{R}}\overline{e_q(-mx)f(xt)}\,e(xk)\,dx}_{x\mapsto x/t}$$

$$= \frac{\tau(\chi)}{tq}\sum_{m(q)}\overbrace{\chi(\overline{m})}^{\overset{\shortparallel}{\chi}(\overline{n-qk}).}\sum_{k\in\mathbb{Z}}\hat{f}\left(\frac{\overbrace{qk-m}^{\ell}}{tq}\right)$$

$$\underbrace{\frac{1}{t}\int_{\mathbb{R}}f(x)\,e\left(\frac{x}{t}\left(k-\frac{m}{q}\right)\right)dx.}_{\hat{f}\left(\frac{qk-m}{tq}\right).}$$

$$\sum_{\ell\in\mathbb{Z}}\hat{f}\left(\frac{\ell}{tq}\right)\chi(-\overline{\ell}).$$

$$2\vartheta_{f,\chi}(t) = \frac{\chi(-1)\tau(\chi)}{tq}\sum_{\ell\in\mathbb{Z}}\overline{\chi}(\ell)\hat{f}\left(\frac{\ell}{tq}\right)$$

$$= 2\frac{\chi(-1)\tau(\chi)}{tq}\vartheta_{\hat{f},\overline{\chi}}\left(\frac{1}{tq}\right).$$

Following Riemann-style:

$$\tilde{\vartheta}_{f,\chi}(s) = \int_0^\infty \underbrace{\vartheta_{f,\chi}(t)}_{\sum_{n\geq 1}\chi(n)f(nt)}\,t^s\,\frac{dt}{t},$$

at $t\to\infty$, rapid decay, $t\to 0$,

(Re $s>1$).

$$= \sum_{n\geq 1}\chi(n)/n^s\int_0^\infty f(nt)\,t^s\,\frac{dt}{t} = L_\chi(s)\,\tilde{f}(s).$$

$$\Rightarrow = \int_T^\infty \theta_{f,\chi}(t)\, t^s \frac{dt}{t} + \int_0^T \frac{\chi(-1)\tau(\chi)}{tq}\, \vartheta_{\hat f,\bar\chi}\left(\frac{1}{tq}\right) t^s \frac{dt}{t}$$

$$u = \frac{1}{tq}, \quad \frac{du}{u} = \frac{-dt}{t}$$

$$= \int_{\frac{1}{Tq}}^\infty \frac{\chi(-1)\tau(\chi)}{q}\, \vartheta_{\hat f,\bar\chi}(u) \left(\frac{1}{uq}\right)^{s-1} \frac{du}{u}$$

$$L_\chi(s)\tilde f(s) = \underbrace{\int_T^\infty \theta_{f,\chi}(t)\, t^s \frac{dt}{t} + \int_{\frac{1}{Tq}}^\infty q^{-s}\chi(-1)\tau(\chi)\, \vartheta_{\hat f,\bar\chi}(t)\, t^{1-s}\frac{dt}{t}.}_{\text{entire}}$$

(true for any $T>0$).

$\Rightarrow L_\chi(s)$ has analytic cont as entire function. (no poles at $s=1,0$! $\chi(0)=0$)

Now assume $\mathrm{Re}(s) \geq 0$, $T \to \infty$,

$$\underbrace{L_\chi(s)\tilde f(s)} = \int_0^\infty q^{-s}\chi(-1)\tau(\chi) \sum_{n\geq 1} \bar\chi(n) \hat f(nt)\, t^{1-s}\frac{dt}{t}$$

$$\text{(FE).} \quad = q^{-s}\chi(-1)\tau(\chi)\sum_{n\geq1}\frac{\bar\chi(n)}{n^{1-s}}\ \widetilde{\hat f}(1-s) = q^{-s}\chi(-1)\tau(\chi)\underbrace{L_{\bar\chi}(1-s)\,\widetilde{\hat f}(1-s)}$$

$$\Rightarrow \quad q^{s/2} L_\chi(s)\, \tilde{f}(s) = q^{\frac{1-s}{2}}\, \chi(-1) \frac{\tau(\chi)}{\sqrt{q}}\, L_{\bar\chi}(1-s)\, \widetilde{\hat{f}}(1-s).$$

For $f = \begin{cases} e^{-\pi t^2} \\ \underline{\underline{t}}\, e^{-\pi t^2} \end{cases}$

$$\Lambda_\chi(s) = \underbrace{\chi(-1)\frac{\tau(\chi)}{\sqrt{q}}}\, \Lambda_{\bar\chi}(1-s).$$

Claim: $|\cdot| = 1$. (root number).

I.e: If χ = primitive $\Rightarrow$ $\underbrace{|\tau(\chi)| = \sqrt{q}.}$

Pf:

Parseval: $\|\chi\|_2^2 = \|\hat\chi\|_2^2$

$$\sum_{r(q)}{}' |\chi(r)|^2 = \varphi(q)$$

$$\|$$

$$\sum_{m(q)}{}' |\underbrace{\hat\chi(m)}_{\chi(\bar m)\, \underbrace{\hat\chi(1)}_{\frac{\tau(\chi)}{\sqrt{q}}}}|^2 = \sum_{m(q)}{}' |\chi(\bar m)|^2 \left|\frac{\tau(\chi)}{\sqrt{q}}\right|^2$$

$$= \frac{|\tau(\chi)|^2}{q}\, \varphi(q)$$

Teaser: ① $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots = \frac{\pi}{4}$

(Leibniz 1690s, Madhava 1400s?) "$L_{\chi_4}(1)$

② $$\frac{1}{1}-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{0}{5}+\frac{1}{6}-\frac{1}{7}-\frac{1}{8}+\frac{1}{9}+\frac{0}{10}+\dots=\frac{2\log\varphi}{\sqrt{5}}$$

$$= L_{\chi_5}(1).$$

$$\left(\varphi=\frac{1+\sqrt{5}}{2}\right).$$

③ $$\frac{1}{1}+\frac{1}{2}-\frac{1}{3}+\frac{1}{4}-\frac{1}{5}-\frac{1}{6}+\frac{0}{7}+\dots=\frac{\pi}{\sqrt{7}}$$

$$= L_{\chi_7}(1).$$

Beilinson: Which L-values are (motivic) "periods"?