Last time: PNT: $\sum_{n<x} \Lambda(n) = x + O(x e^{-c\sqrt{\log x}})$.

$\Downarrow$

Back to Euler: (1737) ∞ly many primes $\Leftarrow \underbrace{\sum_p \frac{1}{p} = \infty}$, $\sum_{p<x} 1 \sim \frac{x}{\log x} \Rightarrow p_n \sim n \log n$.

$\Rightarrow \sum_{p_n} \frac{1}{n \log n} \sim \int_2^\infty \frac{dt}{t \log t} = \log\log t \Big|_2^\infty$ $\quad \log \underbrace{\zeta(1)}$ rate $\to \infty$ at $\log x$

Before Dirichlet Thm: (1837) $\forall (a,q)=1 \ \exists \infty$ly many $p \equiv a(q)$.

$\left(\sum_{p \equiv a(q)} \frac{1}{p} = \infty \Longrightarrow \right)$.

$\downarrow$

Lots of special cases known.

E.g.: $p \equiv 3(4)$. Euclid works: If $p_1, \dots, p_k$ finite list of primes $\equiv 3(4)$, then $N = 4 p_1 \cdots p_k - 1 \equiv 3(4)$, odd, $\prod_{p \equiv 1(4)} p \equiv 1(4)$. $\Rightarrow \exists$ more primes $\equiv 3(4)$.

E.g.: $p \equiv 1(4)$ Fermat: $N = \square + \square \Leftrightarrow N = 2^{e_2} \prod_{p \equiv 1(4)} p^{e_p} \prod_{q \equiv 3(4)} q^{2f_q}$.

So if $p_1, \dots, p_k$ full list of $p_j \equiv 1(4)$, let

$N = (\underbrace{4p_1 \cdots p_k}_{P})^2 + 1^2 = M^2$, $\Rightarrow \cancel{X}$; $\underbrace{P^2 - M^2}_{(P-M)(P+M)} = 1$

Lots of special cases known (Euler, Legendre, Gauss) $\leftarrow \equiv 1 (p)$.

Dirichlet gives complete solution, follow Euler + new ideas

<u>Idea:</u> $\displaystyle \sum_{p \equiv a(q)} \frac{1}{p} \overset{\to\infty}{=} \sum_p \frac{1}{p} \underbrace{\mathbb{1}_{p \equiv a(q)}}_{= \frac{1}{q} \sum_{b(q)} e_q(b(p-a))}$.

$$\boxed{\begin{aligned} e(x) &= e^{2\pi i x} \\ e_q(x) &= e^{2\pi i x/q}. \end{aligned}}$$

<u>Exercise</u> orthogonality of additive characters

"Natural"? object: $\displaystyle \sum_n \frac{e_q(bn)}{n^s}$.

Lerch zeta function

Additive characters: $\chi(a+b) = \chi(a) \cdot \chi(b)$

multiplicative char: $\chi(a \cdot b) = \chi(a) \cdot \chi(b)$.

$\mathbb{Z}/q$, $G = (\mathbb{Z}/q)^\times$ For q prime, Gauss knew that this "group" is cyclic, that is, $\exists$ primitive root

$\Leftrightarrow$ generator, r s.t. $\{r, r^2, r^3, \dots, r^{q-1}\} = (\mathbb{Z}/q)^\times$, $r^{q-1} \equiv 1$ (Fermat)

$\hat{G}$ = "unitary dual" = unitary irred rep $/\sim$.

$\overset{\uparrow}{\text{abel}}$ = 1-D rep's = characters $\chi : G \to \mathbb{C}^\times$ homom.

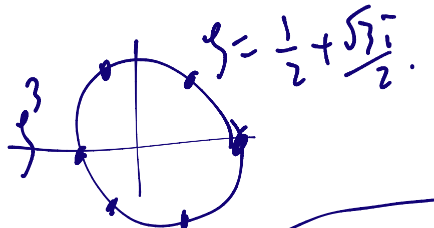
For q prime, need only determine value of $\chi(r)$.

All else is determined. Exercise: $r=2$ is a root for ∞ly many primes?

E.g.: $q=7$, $r \neq 2$ not a root $2^3 \equiv 1\ (q)$.

Is $r=3$ a root? $3, \underset{3^2}{2}, \underset{3^3}{6}, \underset{3^4}{4}, \underset{3^5}{5}, \underset{3^6}{1}$.

$1 = \chi(r^{q-1}) = \chi(r)^{q-1}$



	3	2	6	4	5	1
χ_0	1	1	1	1	1	1
χ_1	ζ	ζ^2	$\cancel{\zeta^3}\ -1$	ζ^4	ζ^5	1
χ_2	ζ^2	ζ^4	1	ζ^2	ζ^4	1
χ_3	-1	1	-1	1	-1	1
χ_4	-	-	-	-	-	1
χ_5	-	-	-	-	-	1

Legendre symbol

$$\left(\frac{n}{7}\right) = \begin{cases} 0 & 7 \mid n \\ 1 & n = \square \\ -1 & n \neq \square. \end{cases}$$

Exercise:

$$\frac{1}{\varphi(q)} \sum_{\chi(q)} \chi(a) = \begin{cases} 1 & a \equiv 1 \\ 0 & \end{cases}$$

$$\frac{1}{\varphi(q)} \sum_{a(q)}{}' \chi(a) = \begin{cases} 1 & \chi = \chi_0 \\ 0 & \text{else} \end{cases}$$

$\sum'_{a(q)} := \sum_{a(q),\ (a,q)=1}$

$\hat{G} \cong (\mathbb{Z}/q)^\times$, $|(\mathbb{Z}/q)^\times| = \varphi(q)$

$\mathbb{Z}/q \supset (\mathbb{Z}/q)^\times$. ring, mult. group

Dirichlet: Extend χ to $: \mathbb{N} \to \mathbb{C}$, by $\chi(n) = 0$ if $(n,q) > 1$.

$(a,q)=1$

$$\sum_p \frac{1}{p} \underbrace{\mathbb{1}_{p \equiv a(q)}}_{\bar{a}p \equiv 1(q)} = \sum_p \frac{1}{p} \frac{1}{\varphi(q)} \sum_{\chi(q)} \underbrace{\chi(\bar{a}p)}_{\chi(\bar{a})\chi(p)}.$$

$a\bar{a} \equiv 1(q)$.

$$= \frac{1}{\varphi(q)} \sum_{\chi(q)} \chi(\bar{a}) \left(\sum_p \frac{\chi(p)}{p} \right)$$

To mimic ~~Euclid~~ Euler, natural to study $\lim_{s \to 1^+}$ ($s \in \mathbb{R}$):

$$\frac{1}{\varphi(q)} \sum_{\chi(q)} \chi(\bar{a}) \left(\sum_p \frac{\chi(p)}{p^s} + \underbrace{\frac{\chi(p^2)}{2p^{2s}} + \frac{\chi(p^3)}{3p^{3s}} + \cdots}_{= O(1) \text{ as } s \to 1^+} \right)$$

$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$

$x = \frac{\chi(p)}{p^s}$.

Caveat: Need to worry about $\log_{\mathbb{C}}$ & branch cuts!

Def: $\ell_\chi(s) := \sum_p \sum_{k\geq 1} \frac{\chi(p)^k}{k p^{ks}}$. $\quad (\mathrm{Re}(s)>1)$.

Obs: $\underbrace{\exp}\left(\ell_\chi(s)\right) = \sum_{n\geq 1} \frac{\chi(n)}{n^s}$.

$\shortparallel$

$1 + \underbrace{\ell_\chi(s)} + \frac{\ell_\chi(s)^2}{2} + \frac{\ell_\chi(s)^3}{3!} + \cdots$

$= 1 + \left(\sum_p \frac{\chi(p)}{p^s} + \frac{\chi(p^2)}{2p^{2s}} + \cdots \right) + \frac{1}{2}\left(\sum_p \frac{\chi(p)}{p^s} + \frac{\chi(p^{2s})}{2p^{2s}} + \cdots\right)^2$

$= 1 + \sum_p \frac{\chi(p)}{p^s} + \sum_p \frac{\chi(p^2)}{p^s}\left(\underbrace{\frac{1}{2}+\frac{1}{2}}\right) + \cdots$

Combinatorial problem: why do all such coefficients $=1$?

$\exp\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots\right)$ ← has same combinatorial problem.

$\frac{\chi(p)}{p^s}$

$= \exp(-\log(1-x)) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$

So prime by prime,

$$\underbrace{\exp(\ell_\chi(s))}_{} = \prod_p \left(\frac{1}{1-\frac{\chi(p)}{p^s}}\right) = \sum_n \frac{\chi(n)}{n^s} \overset{\text{Dirichlet L-function}}{=} L_\chi(s).$$

$\boxed{\operatorname{Re} s > 1}$

Need facts: • $\chi \neq \chi_0$: have analytic cont to entire functions

If $\chi = \chi_0$, $L_{\chi_0}(s) = \prod_{(p,q)=1} \left(1 - \frac{1}{p^s}\right)^{-1} = \underbrace{\zeta(s)}_{\to \infty \text{ as } s \to 1^+} \cdot \underbrace{\prod_{p \mid q} \left(1 - \frac{1}{p^s}\right)}_{}$

has a simple pole; entire.

To prove Dirichlet, can we show that

$$\textcircled{A} = \sum_{p \equiv a(q)} \frac{1}{p^s} + \underbrace{O(1)}_{\text{as } s \to 1^+} = \underbrace{\frac{1}{\varphi(q)} \sum_{\chi(q)} \chi(\bar{a})\, \ell_\chi(s)}_{\mathbb{R}\text{-valued for real } s.} \to \infty \text{ ? as } s \to 1^+.$$

(If χ is complex char $\Rightarrow$ so is $\bar{\chi}$)

One contribution is $\chi = \chi_0$, $\overset{1}{\chi_0(\bar{a})}\, \underbrace{\ell_{\chi_0}(s) \to \infty}_{} \text{ as } \overset{\mathbb{R}}{s} \to 1^+$.

If no other factor $\to -\infty$, then $\circledast$ diverges.

$$\underbrace{\chi(\bar{a})\, l_\chi(s) + \bar{\chi}(\bar{a})\, l_{\bar{\chi}}(s)}_{\to \pm\infty \text{ or } \to -\infty} = 2\,\mathrm{Re}\left(\chi(\bar{a})\, l_\chi(s)\right).$$

$\to \underline{+\infty}$ or $\underline{\to -\infty}$.

For all $\pm$ case, Could have $\mathrm{Im}\, l_\chi(s) \to \infty$?

Only possibility of divergence: $\mathrm{Re}\, l_\chi(s) \to -\infty$.

Claim: can't have $\mathrm{Re}\, l_\chi(s) \to \infty$.

otherwise $\left|\exp(l_\chi(s))\right| = \exp(\mathrm{Re}(l_\chi(s)))$.

$$\underset{\text{as } s \to 1^+}{\Big|} \gg \quad \underbrace{\exp(l_\chi(s))}_{= L_\chi(s)}$$

$L_\chi(s)$ ← regular at $s = 1$.

Only chance is if $\mathrm{Re}\, l_\chi(s) \to -\infty$.

If this, $L_\chi(s) \to 0$ as $s \to 1$.

Therefore, suffices to show, $\forall \chi \neq \chi_0$, $L_\chi(s) \neq 0$ at $s=1$.

Easy: if χ complex, then $L_\chi(s) \neq 0$.

If $L_\chi(s) = 0$, then $\underbrace{L_\chi(s) = (s-1)\,G(s)}_{}$ entire

$L_{\bar\chi}(s) = (s-1)\,\bar{G}(s)$ | If $\exists \chi \in \mathbb{C}$ with $L_\chi(s)=0$

Trick: Set $q=1$. Look at one $\boxed{\chi = \chi_0}$

$$\exp\Big(\cancel{\tfrac{1}{\varphi(q)}} \underbrace{\sum_{\chi(q)} \cancel{\chi(a)}\, \ell_\chi(s)}_{}\Big) = \prod_{\boxed{\chi(q)}} L_\chi(s). \to 0. \ \cancel{\chi}$$

$\vee\!/$

1

$$\underbrace{\sum_{p\equiv 1(q)} \underbrace{\frac{1}{p^s} + \frac{1}{2p^{2s}} + \cdots}_{}}_{} = \underbrace{\frac{1}{(s-1)}\,\overline{G(s)}}_{\chi_0}\ \underbrace{(s-1)\,\tilde{G}(s)\ (s-1)\,\tilde{G}'(s)}_{\chi \quad \bar\chi}$$

$\vee\!/$

0

Only question remains: Does $L_\chi(s) = 0$ for $\chi \in \mathbb{R}$?

I owe you: $\underbrace{L_\chi(s) \neq 0 \quad \chi \in R}$ (Class # formula)

- $L_\chi(s)$ have anal cont as entire functions.

Alternate version using log-derivatives instead of logs?

$$\underbrace{\frac{1}{\varphi(q)} \sum_\chi \chi(\bar{a}) \frac{L'_\chi(s)}{L_\chi}} = \frac{1}{\varphi(q)} \sum_\chi \chi(\bar{a}) \sum_n \frac{\Lambda(n)\chi(n)}{n^s}.$$

$$= \sum_{n \equiv a(q)} \frac{\Lambda(n)}{n^s}.$$

??

$$\exp(\ldots) \text{''} 1 \leq \underbrace{\prod_\chi L_\chi(s)} = \prod_\chi \left(1 + \frac{\chi(2)}{2^s} + \frac{\chi(3)}{3^s} + \cdots\right)$$

$$\zeta(s)^3 \zeta(s+it)^2 \zeta(s-it)^2 \zeta(s+2it) \zeta(s-2it)$$

$1 \cdot 1^3 \oplus (1 \cdot 1^{it})^2 \boxed{+} (1^{-it})^2 \boxed{+} 1 \cdot 1^{2it} \boxed{+} 1 \cdot 1^{-2it}$