Last time: $f \in S_k(\Gamma)$, hol'c $f : \mathbb{H} \to \mathbb{C}$,

$\Gamma = SL_2(\mathbb{Z})$ ← or any $\begin{pmatrix}1&*\\0&1\end{pmatrix} \in \Gamma$. lattice.

$$f\left(\begin{pmatrix}a&b\\c&d\end{pmatrix}\circ z\right) = (cz+d)^k f(z).$$

$$f(z) = \sum_{n\geq 0} a_f(n) q^n, \quad q = e^{2\pi i z}$$

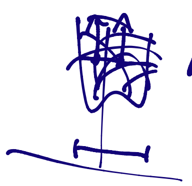
$$\int_0^1 f(x+iy)\, e(-0x)\, dx = a_f(0) = 0. \quad \lim_{y\to\infty} |f|.$$

Showed "trivial" (Hecke) bound: $|a_f(n)| \ll n^{k/2}$

Any exponent less than $\frac{k}{2}$ is called "sub convex". Deligne: $\frac{k}{2} - \frac{1}{2}$.

Long literature ⇝ method of Venkatesh (+ Michel subconvex $GL(2)$ L-fns)

$\boxed{\text{Black Box I:}}$ "effective e.d. of low-lying closed horocycles": $\exists\, \eta > 0$: If $F \in C^\infty \cap L^2(\Gamma\backslash\mathbb{H})$ #f(h)



$n_x = \begin{pmatrix}1&x\\0&1\end{pmatrix}$

$a_y = \begin{pmatrix}\sqrt{y}&0\\0&1/\sqrt{y}\end{pmatrix}$

As $y \to 0$

$$\int_0^1 F(x+iy)\, dx = \frac{1}{\mathrm{vol}(\Gamma\backslash\mathbb{H})} \int_{\Gamma\backslash\mathbb{H}} F \frac{dx\,dy}{y^2} + O_F(y^\eta)$$

$$\int_0^1 F(n_x a_y \cdot i)\, dx$$

pretend $O(\|F\| y^\eta)$

really: Sobolev norm.

Also true for $F \in C^\infty \cap L^2(\Gamma\backslash G)$, $\int_0^1 F(n_x a_y)\, dx = \frac{1}{\mathrm{vol}(\Gamma\backslash G)} \int_{\Gamma\backslash G} F\, dg + O_F(y^\eta)$ Haar

$\mathbb{H} \cong G/K$, $G = SL(2)$,

$$n_x a_y = \begin{pmatrix}1&x\\0&1\end{pmatrix}\begin{pmatrix}\sqrt{y}&0\\0&1/\sqrt{y}\end{pmatrix} = \begin{pmatrix}\sqrt{y}&x/\sqrt{y}\\0&1/\sqrt{y}\end{pmatrix}$$

$$= a_y n_{x/y} = \begin{pmatrix}\sqrt{y}&0\\0&1/\sqrt{y}\end{pmatrix}\begin{pmatrix}1&?\\0&1\end{pmatrix}$$

$?\sqrt{y} = \frac{x}{\sqrt{y}}$

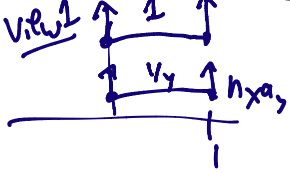
$$= \int_0^1 F(a_y n_{x/y})\, dx \qquad t = \frac{x}{y}$$

$$= \frac{1}{1/y}\int_0^{1/y} F(a_y n_t)\, dt$$

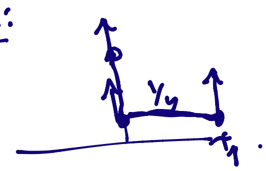
Need effective Ratner for applications in higher rank.

Right regular repn ~ geom action.

view 1:



view 2:



Remark: (Zagier '80) For $\Gamma = SL_2(\mathbb{Z})$, $\eta = \frac{1}{2}$. & $\eta = \frac{3}{4} - \varepsilon \iff$ RH.

For general non-co compact lattices Γ, Sarnak '81

Black Box II: "Effective Mixing of horocycle flow" $\exists\, \eta > 0$, (dep on spect gap.)

$\forall\, F_1, F_2 \in C^\infty \cap L^2(\Gamma\backslash G)$, As $j \to \infty$, $\langle T^j F_1, F_2\rangle = \int_{\Gamma\backslash G} T^j F_1(g)\overline{F_2(g)}\,dg$.

"matrix coefficient"

$(TF)(g) = F(g\cdot n_1)$

Right reg rep, horocycle translation by 1.

asymptotic independence of events

$$= \frac{\langle F_1, 1\rangle\langle 1, F_2\rangle}{\mathrm{vol}(\Gamma\backslash G)} + O_{F_1,F_2}\left((1+|j|)^{-\eta}\right)$$

Howe–Moore Decay of Matrix coefficients, Margulis

$f \in S_k$, Set $\boxed{y = \frac{1}{n}}$: $\int_0^1 f(x + \tfrac{i}{n})\, e(-nx)\,dx = a_f(n)\, e^{-2\pi}$

Need to make automorphic fn't on G out of f.

$$\tilde f\begin{pmatrix} a & b \\ c & d\end{pmatrix} = f\left(\frac{ai+b}{ci+d}\right)(ci+d)^{-k} \in C^\infty \cap L^2(\Gamma\backslash G).$$

(K-isotypic component of discrete series rep'n).

$$f(\underbrace{n_x a_{\frac1n}\cdot i}) = \tilde f(n_x a_{\frac1n})\cdot (n^{k/2}) = n^{k/2}\,\tilde f(a_{\frac1n} n_{nx})$$

$\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt n} & 0 \\ 0 & \sqrt n\end{pmatrix} = \begin{pmatrix} & \\ 0 & \sqrt n\end{pmatrix}$

Hecke: $|\tilde f| < C$, $|a_f(n)| \ll n^{k/2}$.

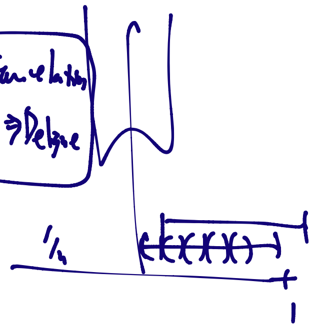
$$\Rightarrow a_f(n) = e^{2\pi} n^{k/2}\int_0^1 \tilde f(a_{1/n} n_{nx})\, e(-nx)\,dx$$

$t = nx$

$$= e^{2\pi} n^{k/2}\frac{1}{n}\int_0^n \tilde f(a_{1/n} n_t)\,\underbrace{e(-t)}\,dt$$

invariant under $t \mapsto t+1$

sqrt cancellation on $\int_0^n$ $\Rightarrow$ Deligne



"amplification" van der Corput "trick":

$J > 0$ parameter to be chosen

$\tilde f(a_{1/n} n_{t+1})$

$$= e^{2\pi} n^{k/2}\frac{1}{nJ}\int_0^n \sum_{j=1}^{J} \overline{T^j \tilde f(a_{1/n} n_t)}\, e(-t)\,dt$$

Cauchy–Schwarz:

$$|a_f(n)|^2 \ll \frac{n^k}{n^2 J^2}\int_0^n \cancel{|e(-t)|^2}\,dt \cdot \int_0^n \left|\sum_{j=1}^{J} T^j \tilde f(a_{1/n} n_t)\right|^2 dt.$$

Let $F(g) := \left|\sum_{j=1}^{J} T^j \tilde{f}(g)\right|^2 \in \text{"}C^\infty\text{"} \cap L^2(\Gamma\backslash G)$.

J will be n^ε.

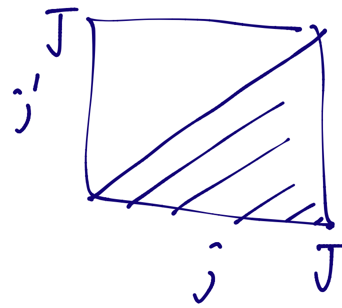
$$|a_f(n)|^2 \ll n^k \frac{1}{n}\frac{1}{J^2}\int_0^n F(a_{1/n} n_t)\,dt. \qquad O_f(J^A n^{-\eta}) \text{ for some } A.$$

$\boxed{\text{BBI}}$
$$= n^k \frac{1}{J^2}\left[\frac{1}{\mathrm{vol}(\Gamma\backslash G)}\int_{\Gamma\backslash G} F(g)\,dg + O_F(n^{-\eta})\right].$$

$$\ll n^k \underbrace{\int_{\Gamma\backslash G}\frac{1}{J^2}\left|\sum_{j=1}^{J} T^j \tilde{f}(g)\right|^2 dg}_{\text{triv bnd } n^k.\ (\text{Hecke})} + \underbrace{O_f(J^A n^{k-\eta})}_{\checkmark}.$$

open square

$$n^k \frac{1}{J^2}\sum_{j=1}^{J}\sum_{j'=1}^{J} \underbrace{\int_{\Gamma\backslash G} T^j \tilde{f}(g)\,\overline{T^{j'}\tilde{f}(g)}\,dg}_{g \mapsto g n_{-j'}}.$$



$$\int_{\Gamma\backslash G} T^{j-j'}\tilde{f}(g)\cdot\overline{\tilde{f}(g)}\,dg.$$

$$\ll n^k \frac{1}{J^2}\left[\sum_{\substack{j=1\\ j'=j}}^{J} \|\tilde{f}\|^2 + J\cdot\sum_{d=1}^{J} \cancel{\sum_{\substack{j=d\\ j'=j-d}}^{J}} \left|\langle T^d \tilde{f}, \tilde{f}\rangle\right|\right].$$

$\boxed{\text{BBII:}}$ Mixing horocycle flow: $\langle T^d\tilde{f}, \tilde{f}\rangle = \cancel{\frac{\langle \tilde{f},1\rangle\langle 1,\tilde{f}\rangle}{\mathrm{vol}(\Gamma\backslash G)}}^{\,0} + O_{\tilde{f}}\left((1+|d|)^{-\eta}\right)$?

Trick: $\langle \tilde{f}, E(\cdot,s)\rangle$ $\left(\mathrm{Res}_{s=1} E(\cdot,s) = \frac{1}{\text{const}}\right)$.

$$= \int_{\mathcal{F}=\Gamma\backslash G} \tilde{f}(g)\, E(g\cdot i, s)\,dg = \int_{\mathcal{F}} \tilde{f}(g) \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \mathrm{Im}(\gamma g\cdot i)^s\,dg.$$

unfold $$= \int_{\Gamma_\infty\backslash G} \tilde{f}(g)\,\mathrm{Im}(g\cdot i)^s\,dg.$$

Iwasawa: $G = \overset{n_x a_y k_\theta}{NAK}$, $dg = \frac{dx\,dy}{y^2}\,d\theta$.

$$= \int_K \int_A \int_{\Gamma_\infty \backslash N} \underbrace{\tilde f(n_x a_y k_\theta)}\; y^s \, dx \, \frac{dy}{y^2}\, d\theta .$$

$$\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} \underbrace{a_y k_\theta} = \begin{pmatrix} * & * \\ c & d\end{pmatrix}$$

indep of x.

$$\int_K \int_A \underbrace{\int_0^1 f(n_x a_y \cdot i)\, dx}_{a_f(0) = 0} \big(\sim y\text{'s } \theta\text{'s} \big)^k \cdot y^s \frac{dy}{y^2}\, d\theta$$

$$\begin{pmatrix}1 & \mathbb{R}\\ 0 & 1\end{pmatrix} = N.$$

$$\Rightarrow \int_{\Gamma\backslash G} \tilde F = 0.$$

$$|a_f(n)|^2 \ll n^k \underbrace{\left[\frac{1}{J^2}\sum_{\substack{j=1\\ j'=j}}^{J} \|\tilde f\|^2 + \frac{1}{J}\sum_{d=1}^{J} \cancel{\sum_{j-j'=d}} |\langle T^d \tilde F, \tilde F\rangle|\right]} + O_f\left(J^A n^{k-\eta}\right).$$

$$\ll n^k \left[\frac{1}{J} + \frac{1}{J^2} J \cdot \underbrace{\sum_{d=1}^{J} d^{-\eta}}_{J^{1-\eta}}\right] + J^A n^{k-\eta}.$$

$$\ll n^k \cdot J^{-\eta} + J^A n^{k-\eta}.$$ choose $J = n^{\varepsilon}$.

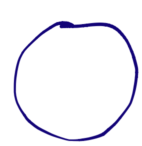
$$\ll n^{k-\eta}.$$

Rmk: E.d. of long closed horocycles & mixing of horocycle flow are "different" dynamical systems, playing against each other

gives application

Furstenberg $\times 2 \times 3$ conjecture.

only (ergodic) measure inv under both $\times 2$ & $\times 3$ is Haar.



$\mathbb{T}$

$A \subset \mathbb{F}_p$, $\max |A+A|, |A \cdot A| \le K \cdot |A|$.

$\Rightarrow |A| > p^{1-\varepsilon}$ or $|A| < p^{\varepsilon}$.

Assume $|A| < p^{1-\varepsilon}$, $|A| > p^{\varepsilon}$.

then

$|A+A| + |A\cdot A| > |A|^{1+\delta}$.