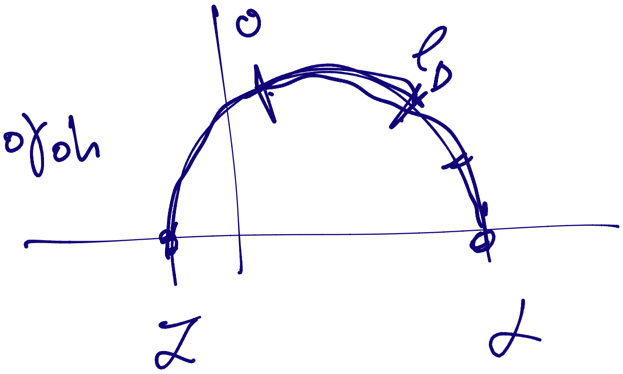
$$\underbrace{\int_0^{\ell_D} \sum_{Q \in \Gamma\backslash X_D} \sum_{\gamma \in \Gamma_\infty\backslash \Gamma/\Gamma_H}}_{\Gamma/\Gamma_H \qquad \begin{pmatrix} * & * \\ c & d \end{pmatrix} \qquad \Gamma_H = \langle M \rangle} \cdot \sum_{h \in \Gamma_H} \cdots Q\circ\gamma\circ h$$



$$\sum_{Q \in \Gamma_\infty\backslash\Gamma} \underbrace{\sum_{h \in \Gamma_H} \int_0^{\ell_D} \cdots h}_{\sum_{n \in \mathbb{Z}} \int_{(n-1)\ell}^{n\ell} \sqcup} \to \int_{\mathbb{R}} \sqcup$$

Hecke operators. Modular forms.

$\Delta = y^2(\partial_{xx} + \partial_{yy}) \circlearrowright L^2(\mathcal{F}^{\mathbb{H}})$.

Cauchy-Riemann operator, $f : \mathbb{H} \to \mathbb{C}$

killed by CR $\Leftrightarrow$ f hol'c, want f to be modular w.r.t. $\Gamma = SL_2(\mathbb{Z})$, of weight k:

<u>Def:</u> f is modular if $\forall\, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $\forall z \in \mathbb{H}$,

$$f(\gamma \circ z) = \underbrace{(cz+d)^k}_{j_{\gamma,k}(z)} f(z).$$

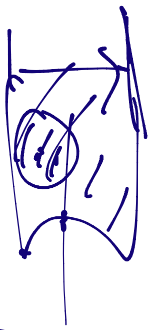
$\boxed{\gamma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ f(z+1) = f(z)}$

$\boxed{\text{Exercise: } j_{\gamma_1\gamma_2}(z) = j_{\gamma_1}(\gamma_2 z) \cdot j_{\gamma_2}(z)}$

Let $M_k = \{ f : \mathbb{H} \to \mathbb{C} :$ hol'c & $\underline{\text{hol'c at } \infty}$ $\}$
$\Updownarrow$ $|f| < C$.

$$f(x+iy) = \sum_{n \in \mathbb{Z}} a_f(n;y)\, e^{2\pi i n x}$$

C-R $\Rightarrow$ $a_f(n;y) = a_f(n) \cdot e^{-2\pi n y}$.



$= \sum_{\substack{\cancel{n\in\mathbb{Z}} \\ n\geq 0}} a_f(n)\, q^n$, where $\underbrace{q = e^{2\pi i z}}$ "chart at ∞".

$z \mapsto e^{2\pi i z} = q$, $|q| = e^{-2\pi y} < 1$

As $y \to \infty$, $|q| \to 0$

$\to a_f(n) = 0\ \forall n < 0$. (Not Laurent series, but power series).

Exercise: Def: holomorphic E-series of wt k:

$$E_k(z) \doteq \sum_{(c,d)=1} \frac{1}{(cz+d)^k} \qquad (k > 2) \Leftrightarrow k \geq 4.$$

$\boxed{\begin{array}{l} k \equiv 1\ (2) \\ \begin{pmatrix}-1 & 0 \\ 0 & -1\end{pmatrix} \in \Gamma \\ f(Iz) = (-1)^k f(z) \\ f(z) = -f(z) \end{array}}$

$$M_k \overset{\text{(Exercise)}}{=} \mathbb{C} E = \overbrace{2\zeta(k)}^{1} + \sum_{n\geq 1} \sigma_{k-1}(n)\, q^n.$$

$\oplus M_k$ = graded ring of all weight

Exercise: E_4 & E_6 generate ring.

Notice: $\overset{1728.}{\ast}\Delta = E_4^3 - E_6^2 \in S_k \subset M_k$ cusp forms $\Leftrightarrow a_f(0) = 0$, $f \to 0$ as $y \to \infty$

Ramanujan discriminant

$$\Delta = q(1-q)^{24}(1-q^2)^{24}(1-q^3)^{24}\cdots = q - 24q^2 + \cdots$$

$$= \sum_{n\geq 1} \tau(n)\, q^n$$

"use η^{24}"

(Euler pentagonal thm or Jacobi triple product) "Ramanujan's tau"

generating series for partition function:

$$\frac{1}{(1-q)(1-q^2)(1-q^3)\cdots} = (1+q+q^2+q^3+\cdots)(1+q^2+q^4+q^6+\cdots)(1+q^3+\cdots)$$

$$= \sum_{n\geq 0} p(n)\, q^n$$

partition function

$3 = 3.$

$3 = 2+1$

$3 = 1+1+1$

If $f \in S_k$, $f(z) = \sum_{n\geq 1} a_f(n)\cdot q^n$,

Hecke L-function (following Riemann memoir).

$$\int_0^\infty f(iy)\, y^s \frac{dy}{y}$$

$$\boxed{q = e^{2\pi i z}}$$

ok if $\mathrm{Re}\, s > \frac{k}{2}+1$.

$$= \int_0^\infty \sum_{n\geq 1} a_f(n)\, e^{-2\pi n y}\, y^s \frac{dy}{y}.$$

$$= \sum_{n\geq 1} \frac{a_f(n)}{(2\pi n)^s} \underbrace{\int_0^\infty e^{-\cancel{2\pi n}y}\, y^s \frac{dy}{y}}_{``\Gamma(s)"}$$

$y \mapsto \frac{y}{2\pi n}$

$$= \Lambda_f(s) = (2\pi)^{-s}\,\Gamma(s)\, L_f(s), \quad \leftarrow = \sum_{n\geq 1} \frac{a_f(n)}{n^s}.$$

Mellin transform converts Fourier expansions into Dirichlet series with same coefficients

Hecke "trivial" bound on $a_f(n)$:

Observe that $\left|\underbrace{f(z)}_{\text{decays exp.}}\ \underbrace{y^{k/2}}_{\text{poly growth}}\right| \to 0$ at ∞, automorphic.

$\forall z \in \mathbb{H}$.

$\Rightarrow |f(z) y^{k/2}| < C = C(f)$.

$$\left|\int_0^1 f(x+iy)\, y^{k/2}\, e(-nx)\, dx\right| = \left|a_f(n)\, y^{k/2}\, e^{-2\pi n y}\right|.$$

$$\le \int_0^1 \left| f(x+iy)\, y^{k/2} \right| dx \le C.$$

$$|a_f(n)| \le C \cdot e^{2\pi n y}\, y^{-k/2}.$$

($L_f(s)$ wouldn't converge anywhere)

But true for all y, let $y = \frac{1}{n}$. $\Rightarrow \ll n^{k/2}$.

Deligne: $|a_f(n)| \ll_\varepsilon n^{k/2 - 1/2 + \varepsilon}$ Ramanujan conj.

What about analytic cont?

$$\int_0^\infty = \underbrace{\int_\alpha^\infty}_{\text{entire}} + \int_0^\alpha f(iy)\, y^s \frac{dy}{y}.$$

Let $\alpha_f(n) := a_f(n) / n^{\frac{k-1}{2}}$.

Deligne: $|\alpha_f(n)| \ll n^{\varepsilon}$

New def: $L_f(s) := \sum_{n\ge 1} \frac{\alpha_f(n)}{n^s} = \sum_{n \ge 1} \frac{a_f(n)}{n^{s + \frac{(k-1)}{2}}}$

FE: $f(z) = \sum_n \alpha_f(n) \cdot n^{\frac{k-1}{2}} q^n$

Consider: $\displaystyle\int_0^\infty f(iy)\, y^{s+\frac{(k-1)}{2}} \frac{dy}{y}$

$$= L_f(s)\,(2\pi)^{-s-\frac{(k-1)}{2}}\,\Gamma\left(s+\frac{k-1}{2}\right) =: \Lambda_f(s).$$

$$= \int_\alpha^\infty + \underbrace{\int_0^\alpha f(iy)\, y^{s+\frac{k-1}{2}} \frac{dy}{y}}_{y \mapsto 1/y}.$$

Rmk: Instead of Poisson sum for theta, use modularity

$$\int_{1/\alpha}^\infty \underbrace{f\left(\frac{i}{y}\right)}_{\shortparallel} y^{-s-\frac{(k-1)}{2}} \frac{dy}{y}.$$

$$f\left(\frac{-1}{iy}\right) = f\Big(\underset{\shortparallel}{\overset{\begin{pmatrix}0&-1\\1&0\end{pmatrix}}{S}}\cdot(iy)\Big) = (iy)^k f(iy)$$

$$= i^k \underbrace{\int_{1/\alpha}^\infty f(iy)\, y^{1-s+\frac{(k-1)}{2}} \frac{dy}{y}}_{\text{entire in } s}. \qquad \alpha \to \infty$$

$$= i^k \Lambda_f(1-s) =: \Lambda_f(s).$$

"Should" we consider these "L-functions"?

In general, $h = \alpha \cdot f + \beta \cdot g \in S_k$,

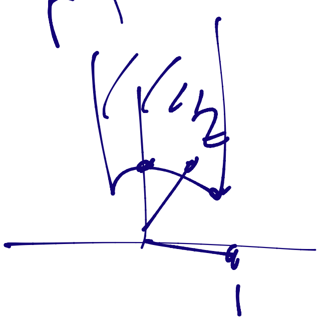
$L_h(s) = \alpha L_f(s) + \beta L_g(s)$ can be zero in region of absolute convergence $s = 100k$.

Need Euler products (Ramanujan Conj! 1917)

Mordell proved 1918, Hecke 30's operators.

Fact: $\dim S_{12} = 1$. Exercise: $\dim S_k = \dim M_k - 1$.

$\Gamma \backslash \mathbb{H} \cong$ space of unimodular lattices



What about index $n \geq 1$ sublattices?

i.e.: $\Gamma \circlearrowright T_n = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_2(\mathbb{Z}) : AD - BC = n \right\}$.

$\gamma \in \Gamma\backslash\Gamma_n$ coset space: $\begin{pmatrix}1 & \ell\\ 0 & 1\end{pmatrix}\begin{pmatrix}a & d\\ c & d\end{pmatrix}\begin{pmatrix}A & B\\ C & D\end{pmatrix} = \begin{pmatrix} & \\ \underset{\underset{0}{\uparrow}}{Ac+Cd} & \end{pmatrix}$

$(A,C)\neq(0,0)$. $\exists\, (c,d)=1$ s.t.

$\alpha\cdot\delta = n$, $\beta \bmod \delta$. $\qquad \underline{\begin{pmatrix}1 & \ell\\ 0 & 1\end{pmatrix}\begin{pmatrix}\alpha & \beta\\ 0 & \delta\end{pmatrix}}$

<u>Lemma:</u> $\boxed{\Gamma\backslash\Gamma_n = \left\{\begin{pmatrix}\alpha & \beta\\ 0 & \delta\end{pmatrix} \;\middle|\; \begin{matrix}\alpha\cdot\delta = n\\ \beta \bmod \delta\end{matrix}\right\}}$ $\begin{pmatrix}\alpha & \beta+\ell\delta\\ 0 & \delta\end{pmatrix}$.

<u>Def:</u> $f|_\gamma(z) = (\det\gamma)^{k/2}(cz+d)^{-k} f(z)$.

$f\in S_k \iff \forall \gamma\in\Gamma,\ f|_\gamma = f.$ $\qquad \boxed{f|_{\gamma_1\gamma_2} = f|_{\gamma_1}|_{\gamma_2}}$

$$(T_n f) = \sum_{\gamma\in\Gamma\backslash\Gamma_n} f|_\gamma = \sum_{\alpha\delta=n}\sum_{\beta(\delta)}\left(\frac{\alpha}{\delta}\right)^{k/2} f\left(\frac{\alpha z+\beta}{\delta}\right)$$

<u>Exercise:</u> $f\in S_k \Rightarrow T_n f \in S_k$. $\leftarrow$ slashing with element of Γ <u>permutes</u> cosets.

<u>Exercise</u>: $T_n \cdot T_m = \sum_{\substack{d \mid n \\ d \mid m}} T_{\frac{nm}{d^2}}$.

$\Rightarrow$ Operators <u>Commute</u>.

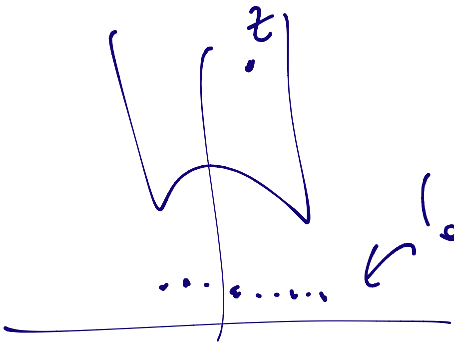
S_k has Hilbert space structure wrt

Petersson inner product: $\langle f, g \rangle_{\Gamma} = \int_{\Gamma \backslash H} \underbrace{f(z)\, \overline{g(z)}\, y^k}_{\text{automorphic}} \frac{dx\,dy}{y^2}$

<u>Exercise</u>: T_n <u>self-adjoint</u> $\langle T_n \cdot f, g \rangle = \langle f, T_n g \rangle$

<u>E.g.</u>: $T_p \cdot f(z) = \sum_{\substack{a=p \\ d=1}} \sum_{b(1)} p^{k/2} f(pz) + \sum_{\substack{a=1 \\ d=p}} \sum_{b(p)} p^{-k/2} f\left(\frac{z+b}{p}\right)$

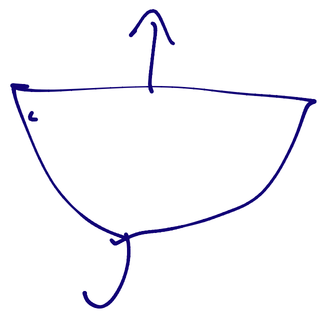
$\cdot pz$

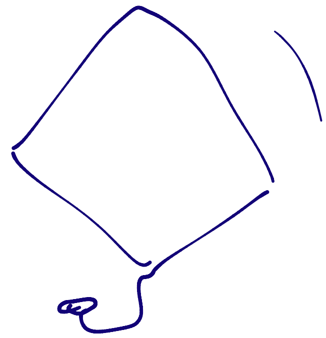


$\leftarrow$ underlying horocycle.

Commuting + self adjoint on fin. dim vect space

$\Rightarrow \exists$ basis of simultaneous Eigenfunctions!!





Do any T_n's kill Δ? Unknown!

Lehmer Conj: $\tau(n) \neq 0$. don't

Say f (Hecke) eigenfunct of all T_n's.

$$\lambda(n)\cdot f = T_n f = \sum_{ad=n}\sum_{b(d)} \left(\frac{a}{d}\right)^{k/2} f\left(\frac{az+b}{d}\right)$$

$$\lambda(n)\sum_{l\geq 1}\alpha_f(l)\, q^l\, l^{\frac{k-1}{2}} = \sum_{ad=n}\left(\frac{a}{d}\right)^{k/2}\sum_{b(d)}\sum_{r\geq 1}\alpha_f(r)\, r^{\frac{k-1}{2}}\, e^{2\pi i r\left(\frac{az+b}{d}\right)}$$

$$= \begin{cases} d, & r\equiv 0\ (d) \quad \cancel{d\mid r} \\ 0, & \text{else.}\end{cases}$$

$$= \sum_{ad=n}\left(\frac{a}{d}\right)^{k/2}\cdot d \sum_{r\geq 1}\alpha_f(r\cdot d)(r\cdot d)^{\frac{k-1}{2}} e^{2\pi i \underbrace{ra}_{=\,l} z}$$

$\boxed{r \mapsto r\cdot d}$

$$= \sum_{l \geq 1} q^{l} \sum_{\substack{a \mid l \\ ad = n}} \left(\frac{a}{d}\right)^{k/2} d \; \alpha_f\left(\frac{l \cdot d}{a}\right) \left(\frac{l \cdot d}{a}\right)^{\frac{k-1}{2}}$$

$l = r \cdot a$

$dr = \frac{ld}{a}$

$\Rightarrow \forall l,$

$$\lambda(n)\, \alpha_f(l)\, \cancel{l^{\frac{k-1}{2}}} = \sum_{\substack{a \mid l \\ a \cdot d = n}} \alpha_f\left(\frac{l \cdot d}{a}\right) a^{1/2} d^{1/2} \cancel{l^{\frac{k-1}{2}}}.$$

$$\Rightarrow \quad \lambda(n)\, \alpha_f(l) = n^{1/2} \sum_{\substack{a \mid l \\ a \cdot d = n}} \alpha_f\left(\frac{l \cdot d}{a}\right).$$

Cor: If $(l, n) = 1$, $\lambda(n)\, \alpha_f(l) = n^{1/2} \alpha_f(l \cdot n)$.

- If $l = 1$, $\lambda(n)\, \alpha_f(1) = n^{1/2} \alpha_f(n)$.

$\Rightarrow$ if $\alpha_f(1) = 0$, $\Rightarrow \forall n \; \alpha_f(n) = 0 \Rightarrow f \equiv 0$.

OK $\alpha_f(1) \neq 0 \Rightarrow$ rescale f to be "Hecke normalized" $\alpha_f(1) = 1$. Then $\dfrac{\lambda(n)}{n^{1/2}} = \alpha_f(n)$.

<u>Exercise</u>: $\Delta^2 \in S_{24}$ ← <u>not</u> eigenfunction for all T_n. what is this in terms of E_4 & E_6??

<u>Cor</u>: Assume f Hecke-normalized. Then

$$(n,l) = 1 \Rightarrow \alpha_f(n)\alpha_f(l) = \alpha_f(n \cdot l).$$

<u>multiplicativity</u>. (<u>Not</u> complete!).

<u>Cor</u>: For $n = p$, $l = p^r$

$$\underbrace{\cancel{\lambda(p)}}_{\cancel{p^{1/2}}\,\alpha_f(p)} \cdot \alpha_f(p^r) = \cancel{p^{1/2}} \sum_{\substack{a \mid p^r \\ a\cdot d = p}} \alpha_f\left(\frac{p^r \cdot d}{a}\right)$$

$$\circledast \quad \boxed{\alpha_f(p)\cdot\alpha_f(p^r) = \alpha_f(p^{r+1}) + \alpha_f(p^{r-1})}.$$

$$L(f,s) = \sum_{n\geq 1} \frac{\alpha_f(n)}{n^s} = \prod_p \left(1 + \frac{\alpha_f(p)}{p^s} + \frac{\alpha_f(p^2)}{p^{2s}} + \dots\right)$$

Exercise from B

$$= \prod_p \left(1 - \frac{\alpha_f(p)}{p} + \frac{1}{p^{2s}}\right)^{-1}.$$

$$1 - \alpha(p)X + X^2 = (1 - X\theta)(1 - \beta X).$$

$$= \prod_p \left(1 - \frac{\theta(p)}{p^s}\right)^{-1}\left(1 - \frac{\beta(p)}{p^s}\right)^{-1}$$

$$\theta(p)\cdot\beta(p) = 1$$

& $\theta(p) + \beta(p) = \alpha_f(p)$.

either θ & β real & $\alpha \geq 2$.

or $\theta = \bar{\beta}$, $|\theta| = |\beta| = 1$, $\underbrace{|\theta + \beta|}_{= |\alpha|} \leq 2$

Deligne's $\rightarrow$

Ref for weights