Recall: $\sum_{p \equiv a(q)} \frac{1}{p} \to \infty$ as long as $L(1,\chi_q) \neq 0$ $\forall \chi$.

Dirichlet Class # Form $\longrightarrow$ $\boxed{(1)}$*

When $q \equiv 3(4)$, $D = -q \equiv 1(4)$, $\boxed{L(1,\chi) \doteq \frac{h(-q)}{\pi\sqrt{q}}}$, when $\boxed{q \equiv 1(4)}$: $\frac{h(q)\log \varepsilon_q}{\sqrt{q}}$, $D = q \equiv 1(4)$.

$\varepsilon_D = \frac{t + \sqrt{D}\, s}{2}$ where $t^2 - Ds^2 = 4$ is minimal Pell solution.

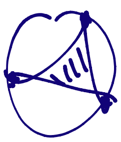
$\boxed{D<0}$

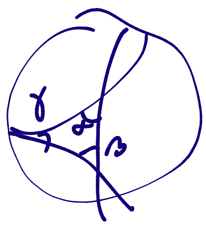
Recall: $E(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma z)^s$ $\quad (\mathrm{Re}\, s > 1)$.

$\leftarrow$ Simple pole at $s=1$.

$\mathrm{Res}_{s=1} E(z,s) = \frac{1}{\mathrm{vol}(\Gamma \backslash \mathbb{H})} = \frac{3}{\pi}$ indep of z!

(Gauss-Bonnet:



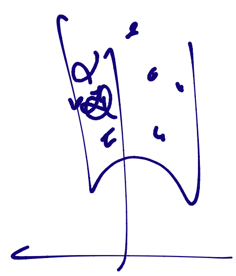


Area $= \pi - \alpha - \beta - \gamma$).

$SL_2(\mathbb{Z}) = \Gamma \circlearrowright X_D = \{ Q = [A, B, C] : B^2 - 4AC = D,\ (A,B,C) = 1 \}$ $\longrightarrow$ $\alpha_Q = \frac{-B + \sqrt{D}}{2A} \in \mathbb{H}$.

$\mathcal{C}_D = \Gamma \backslash X_D$. $\quad h(D) = \# \mathcal{C}_D$

Look at: $\sum_{[Q] \in \mathcal{C}_D} E(\alpha_Q, s) \longleftarrow \mathrm{Res}_{s=1} = \frac{3}{\pi} \cdot h(D)$



$$\sum_{Q \in \Gamma\backslash X_D} \cdot \sum_{\gamma \in \Gamma_\infty\backslash\Gamma} \underbrace{\overbrace{\mathrm{Im}\left(\gamma\cdot\alpha_Q\right)^s}^{\mathrm{Im}(\alpha_{Q\circ\gamma})^s}} \to \left(\frac{\sqrt{D}}{2A}\right)^s \cdot \underbrace{\frac{1}{|c\alpha_Q+d|^{2s}}}_{\frac{1}{Q(c,d)^s}}$$

$$\boxed{\mathrm{Im}\, gz = \frac{\mathrm{Im} z\cdot \det g}{|cz+d|^2}}$$

$$\underbrace{\sqrt{D}^s \sum_{Q\in\Gamma\backslash X_D} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma}}_{\text{unfold}} \frac{1}{Q(c,d)^s}$$

|| Epstein zeta function for Q.

$(Q\circ\gamma)(0,1)$.

$$= \sqrt{D}^s \sum_{Q\in\Gamma_\infty\backslash X_D} \frac{1}{Q(0,1)^s}$$

$$\boxed{\Gamma_\infty\backslash X_D = \{(A,B,C) : B^2-4AC=D,\ B \bmod 2C\}},$$

$$= \sqrt{D}^s \underbrace{\sum_{C\geq 1} \frac{1}{C^s}\cdot R(C)}_{\underbrace{\sum_{C\geq 1}\frac{1}{C^s}\sum_{\ell | C}\chi(\ell)}_{\zeta(s)\,L(s,\chi)}},\qquad R(C) = \#\{B \bmod 2C : \exists A,\ B^2-4AC=D\} = \sum_{\ell|C}\chi(\ell).$$

$$\left(=\zeta_K(s),\ K=\mathbb{Q}(\sqrt{D})\right).$$

$$\mathrm{Res}_{s=1} = \sqrt{D}\,L(1,\chi). \Rightarrow L(1,\chi) = \frac{h(D)}{\pi\sqrt{D}}. \quad (\triangle \text{ fix constant})$$

For $D>0$: $\alpha_Q = \frac{-B+\sqrt{D}}{2A}, \bar{\alpha}$

$$A = \sum \int E(z, s)\, d\,\text{arclength}$$

$Q \in \Gamma \backslash X_D$ $\quad C_Q \leftarrow$ length ℓ_Q.

For Q disc D, hyperbolic matrix $M_Q = M \in \Gamma$

$$= \begin{pmatrix} \frac{t-Bv}{2} & -Cv \\ Av & \frac{t+Bv}{2} \end{pmatrix}$$

$$\boxed{t^2 - Dv^2 = 4 \quad \text{find solution}}$$

If $g \in G \cong T^1\mathbb{H}$, is a starting pt of a closed geod, then $g a_t \big|_{t=\ell} = Mg$ so the closed geod corr to M "starts" at $\boxed{g = \text{e-vects of } M}$ ($\sim g_Q$) & has length ℓ, where $a_t = \begin{pmatrix} e^{t/2} & \\ & e^{-t/2} \end{pmatrix}$

$\operatorname{tr} M = t = 2\cosh \ell/2 = e^{\ell/2} + e^{-\ell/2}$

$$e^{\ell} - e^{\ell/2} t + 1 = 0, \quad e^{\ell/2} = \frac{t+\sqrt{t^2-4}}{2} = \frac{t+\sqrt{D}v}{2} = \varepsilon_D$$

$$\Rightarrow \ell_Q = 2\log \varepsilon_D = \ell_D. \qquad \operatorname*{Res}_{s=1} \cancel{\Phi} = \underbrace{h(D)\cdot \log \varepsilon_D}\cdot\frac{6}{\pi}.$$

$$\overset{\text{tole}}{\Phi}(s) = \sum_{Q\in_T \backslash X_D} \int_0^{\ell_D} E(\underbrace{g_Q\cdot a_u\cdot i}, s)\, du.$$

If $M = \begin{pmatrix} a & b \\ c & d\end{pmatrix}$, eigenvalues $\lambda, \frac{1}{\lambda} = \bar\lambda$,

$$\boxed{\begin{aligned} \lambda &= \frac{t+\sqrt{t^2-4}}{2} \\ &= e^{\ell/2} = \varepsilon_D \end{aligned}}$$

$$|M-\lambda I| = \begin{vmatrix} a-\lambda & b \\ c & d-\lambda\end{vmatrix} = \overset{1}{\cancel{ad}} - t\lambda + \lambda^2 - \cancel{bc} = 0$$

$$0 = (M-\lambda)v = \begin{pmatrix} a-\lambda & b \\ c & d-\lambda\end{pmatrix}\begin{pmatrix}\lambda-d \\ c\end{pmatrix} \leftarrow \text{eigenvector},$$

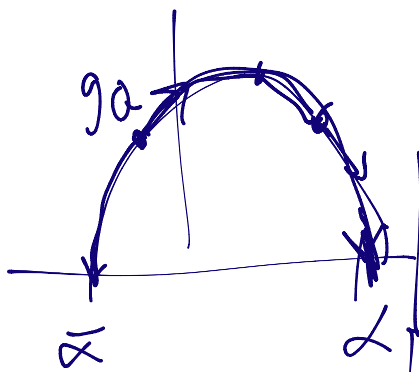
$$g = \begin{pmatrix} \lambda-d & \bar\lambda-d \\ c & c\end{pmatrix}, \in GL_2^+(\mathbb{R}).$$

$$\boxed{\text{Exercise: } g\, a_\ell\, g^{-1} = M.}$$

$$M_Q = \begin{pmatrix} \frac{t-B_v}{2} & -C_v \\ A_v & \frac{t+B_v}{2}\end{pmatrix}, \quad g_Q = \frac{1}{A^{\frac{1}{v}}}\begin{pmatrix} \frac{t+\sqrt{D}v}{2} - \left(\frac{t+B_v}{2}\right) & \frac{t-\sqrt{D}v}{2} - (\dots) \\ A_v & A_v\end{pmatrix}$$

$$t^2 - D_v^2 = 4.$$

$$g_Q = \begin{pmatrix} \frac{-B+\sqrt{D}}{2A} & \frac{-B-\sqrt{D}}{2A} \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} \alpha & \bar{\alpha} \\ 1 & 1 \end{pmatrix}.$$



$$g_Q \underbrace{a_t \cdot i}_{e^t \cdot i} = \frac{\alpha e^t \cdot i + \bar{\alpha}}{1 \cdot e^t \cdot i + 1} \Bigg|_{t \to \infty} \to \alpha$$

$$t \to -\infty \to \bar{\alpha}.$$

$$\boxed{\alpha_{Q\circ\gamma^{-1}} = \gamma \cdot \alpha_Q.}$$

$$\boxed{Q\circ\gamma^{-1},\ g_{Q\circ\gamma^{-1}} = \begin{pmatrix} \gamma\alpha & \gamma\bar{\alpha} \\ 1 & 1 \end{pmatrix}}$$

$$\Phi(s) = \int_0^{l_D} \sum_{Q \in \Gamma \backslash X_D} \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \mathrm{Im}(\gamma \cdot g_Q a_u \circ i)^s \, du$$

$$\gamma = \begin{pmatrix} * & * \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} * & * \\ c & d \end{pmatrix}\begin{pmatrix} \alpha & \bar{\alpha} \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} * & * \\ c\alpha+d & c\bar{\alpha}+d \end{pmatrix}$$

$$\left(\frac{e^u \det g}{|(c\alpha+d) e^u i + (c\bar{\alpha}+d)|^2} \right)^s$$

$$\det g = \alpha - \bar{\alpha}$$

$$= \frac{-B+\sqrt{D}}{2A} - \left(\frac{-B-\sqrt{D}}{2A}\right)$$

$$= \frac{\sqrt{D}}{A}.$$

$$\boxed{Q(x,y) = A(x-\alpha y)(x-\bar{\alpha} y).}$$

$$\sqrt{D}^{\,s} \sum_{Q \in \Gamma \backslash X_D} \sum_{(c,d)=1} \frac{1}{A^s} \int_0^{l_D} \left(\frac{1}{(c\alpha+d)^2 e^u + (c\bar{\alpha}+d)^2 e^{-u}} \right)^s du. \Bigg|_{(c,d)^{(q)}}$$

$$\hookrightarrow = \sqrt{D}^{s} \sum_{Q \in \Gamma_\infty \backslash X_D} \frac{1}{A^s} \underbrace{\int_0^{\ell_D} \left(\frac{1}{e^u + e^{-u}}\right)^s du}_{} \to I_D(s) \text{ pulls out}$$

$$X_D = \{(A,B,C) : B^2 - 4AC = D,\ B \in \mathbb{Z}/2A\}.$$

$$= \sqrt{D}^{s} I_D(s) \cdot \underbrace{\zeta(s) L(s,\chi)}_{\zeta_K(s)} \to \boxed{\begin{matrix}\sqrt{D} \cdot L(1,\chi) \\ \shortparallel \\ \cdot\, h(D) \log \varepsilon_D\end{matrix}}$$

$$\boxed{\int_1^{\varepsilon_D} \frac{1}{(u+u^{-1})^s} \frac{du}{u} = \int_{\varepsilon_D}^{\varepsilon_D^2} \frac{1}{(\varepsilon v + \frac{1}{\varepsilon} u^{-1})^s} \frac{du}{u}}$$

$$\boxed{\underline{L(1,\chi)} = \frac{h(D) \log \varepsilon_D}{\sqrt{D}}}$$

<u>Exercise:</u> Fix this

$$\sum_{n \in \mathbb{Z}} \int_{(n-1)\ell_D}^{n\ell_D} \frac{1}{(e^u + e^{-u})^s} du = \int_{-\infty}^{\infty} \frac{1}{(e^u + e^{-u})^s} du$$

$$= \frac{\pi}{2} \ldots$$