$\min_{k\in\mathbb{Z}} \left| \frac{a z_1}{q} - \psi + k \right| = \left\| \frac{a \cdot z_1}{q} - \psi \right\| < \frac{1}{X}.$ $\left(\frac{a}{q} z_1\right)$

$\| \theta z - \theta' z' \| < \frac{1}{X}.$

$v \mid z_1$, $v = (z_1, q)$. $b = z_1 / v$.

$$\boxed{\left\| \frac{a \cdot b}{r} - \psi \right\| < \frac{1}{X}.}$$

$\#\{\text{such within } \frac{1}{X} \text{ of } \psi\} < \underbrace{\frac{1}{X} \cdot \frac{Q^2}{v^2} + 1}$

Summary:

$$\sum_{\substack{z, z' \\ z = 0}} \cdots \le \sum_{z = (z_1, z_2)} \sum_{z' = z} \sum_{\substack{v \mid z_1 \\ v < Q}} \sum_{\substack{q' \le Q \\ q' \equiv 0 (v')}} \sum_{a' (q'), \psi = \frac{a'}{q'} z_1}' \sum_{\substack{f \in \mathcal{F} \\ r = \mathrm{denom}(f) \\ q = r \cdot v}}' \sum_{\substack{a (q) \\ a \text{ determined}}}' 1$$

$$\le \sum_{z} \sum_{z' = z} \sum_{\substack{v' \mid z_1 \\ v' < Q \\ v' \ge v}} \frac{Q}{\cancel{v'}} \, Q \left(\frac{1}{X} \frac{Q^2}{\cancel{v^2}} + 1 \right) \cancel{v} \ldots$$

Conjecture (McMullen '12) Classical Arithm Chaos:

$[a_0, \ldots, a_j, \overline{a_{j+1}, \ldots, a_{j+\ell}}] = a_0 + \frac{1}{a_1 \ldots}$ repeat

(Babylonians / Lagrange)

Exercise: α is a quad irrat $\Leftrightarrow$ ctd frac expansion of α is eventually periodic.

Conj: $\exists\, c > 1$ s.t.

$$\# \left\{ [a_0, a_1, \dots, a_\ell] \in \mathbb{Q}(\sqrt{5}) \;\middle|\; 1 \le a_i \le 2 \right\} \gtrsim c^\ell \quad \text{as } \ell \to \infty.$$

"pos entropy".

$\# \to \infty$ not known!

without $\longrightarrow \# = 2^{\ell+1}$

$\ell \approx 20+$
$\# \sim 8$?

More generally, fix $K = \mathbb{Q}(\sqrt{D})$, $D > 0$ not square.

Fix $\mathcal{A} \subseteq \mathbb{N}$ s.t. $\delta_{\mathcal{A}} > 1/2$. Then $\exists\, c = c(K, \mathcal{A}) > 1$

s.t.

$$\# \left\{ [a_0, a_1, \dots a_\ell] \in K \;\middle|\; a_i \in \mathcal{A} \right\} > c^\ell.$$

Local-Global Conj (Bourgain-K): For $\mathcal{A} = \{1, 2\}$, $(\delta > 1/2)$

$\Gamma = \Gamma_{\mathcal{A}} = \Gamma_2 = \left\langle \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix}, a = 1 \text{ or } 2 \right\rangle^+$, let f

be a nonconstant linear map $f : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \mapsto aA + bB + cC + dD \in \mathbb{Z}$.

Call n admissible if $n \in f(\Gamma) \pmod q$, $\forall q$,

Then for $n \asymp N$, & Γ, $(N \to \infty)$, $\#\{\gamma \in \Gamma \cap B_N \mid f(\gamma) = n\}$

$\boxed{\text{Lemma: Loc-Glob with fctr} \Rightarrow \text{McMullen.}}$ $> \dfrac{N^{2\delta}}{N} N^{-o(1)}$.

Before:
$$\begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_\ell & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d & * \\ b & * \end{pmatrix}$$
$$\Updownarrow$$
$$\frac{b}{d} = [0, a_1, a_2, \ldots, a_\ell].$$

Exercise:
$$M = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_\ell & 1 \\ 1 & 0 \end{pmatrix}. \quad (\ell+1 \text{ even}).$$

If $M\alpha = \alpha \Rightarrow \alpha \in K = \mathbb{Q}(\sqrt{D})$, $D = \mathrm{tr}^2 M - 4 > 0$.

$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. $\quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \alpha = \alpha$.

$\overset{?}{=} \dfrac{a\alpha + b}{c\alpha + d}$, $\quad \alpha = \dfrac{\cdots \pm \sqrt{(a+d)^2 - 4(ad - bc)}}{\cdots}$

$= [\overline{a_0, a_1, \ldots, a_\ell}]$.

So to understand when $[\overline{a_0,\dots,a_\ell}] \in \mathbb{Q}(\sqrt{D})$,

need to understand when $\mathrm{tr}^2 M - 4 = D\cdot s^2$.

So $t = \mathrm{tr}\, M$, then $t^2 - Ds^2 = 4$. $\boxed{\left(\frac{t}{s}\right)^2 - D = \frac{4}{s^2}.}$

$t^2 - Ds^2 \to$ binary quadratic, which #s does it rep?

$D > 0$, $D \neq \square$, rep's 1. $t^2 - Ds^2 = 1$.

Q: For which D does $t^2 - Ds^2 = -1$ have sols?

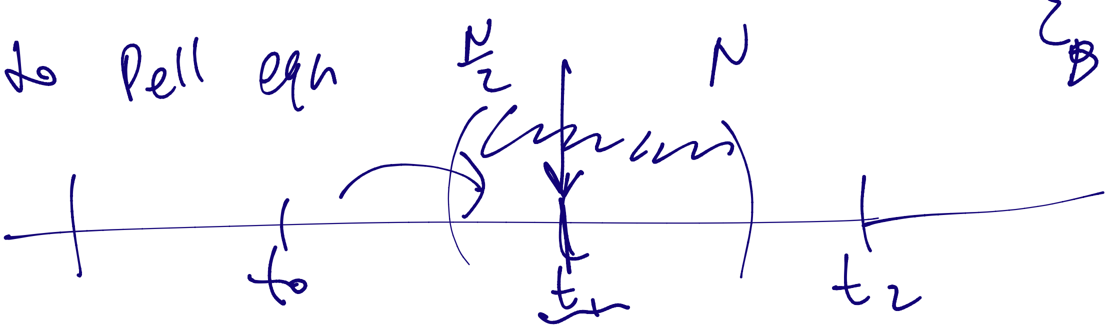
Unit group of $\mathcal{O} \subseteq K = \mathbb{Q}(\sqrt{D})$,

$\varepsilon_0 = \frac{t_0 + \sqrt{D}\, s_0}{2}$ ~) least t_0, s_0 solving

$N(\varepsilon_0) = \left(\frac{t_0 + \sqrt{D}\, s_0}{2}\right)\left(\frac{t_0 - \sqrt{D}\, s_0}{2}\right) = \frac{t_0^2 - Ds_0^2}{4} = 1$.

$\mathcal{O}_K^\times = \langle \pm \varepsilon_0 \rangle$

Solutions to Pell eqn (t_0, s_0). $\varepsilon_0 > 1$.



In a linear interval, $\exists \asymp 1$ value of $t \asymp N$. bc-glod $\Rightarrow \#\{\gamma \in \Gamma \cap B_\rho \text{ s.t. } \operatorname{tr}\gamma = t\} \gg N^{2\eta_0}$.

$\ell(\gamma) \asymp \log N$.

$\Rightarrow \#\{\gamma \in \Gamma \cap \underbrace{WB_\ell}_{\text{wordlength ball}} \mid \alpha_\gamma \in K\} \gg e^{\eta \ell} = c^\ell$.

Exercise: If $\gamma = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_\ell & 1 \\ 1 & 0 \end{pmatrix}$, then $\log \|\gamma\| \asymp \ell$.