Last time: Poisson Summation $f:\mathbb{R}\to\mathbb{R}$ nice

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{m\in \mathrm{Spec}(\Delta \circlearrowright L^2(\Gamma\backslash G))} \hat{f}(m).$$

∂_{xx} $G=\mathbb{R}$, $\Gamma=\mathbb{Z}$.

Review of pf:

- point-pair invariant $\forall u\in\mathbb{R}$.

$$k(x,y) = k(\underbrace{x+u}_{u+x},\underbrace{y+u}_{u+y}) = f(\underbrace{y-x}_{-x+y}).$$

right-action $\quad = f(\underbrace{y+u-(x+u)}_{\cancel{u}y-x}).$

right-regular rep $(\pi(u).f)(x) = f(x+u).$

- automorphize: (invariant under Γ) $K(x,y) = \sum_{\gamma\in\Gamma} k(\gamma+x,y)$ on $\Gamma\backslash G \times \Gamma\backslash G$.

- Make "Hilbert-Schmidt" operator from this "kernel":

$$I_K \circlearrowright \mathcal{H} = L^2(\Gamma\backslash G),\quad (I_K g)(x) = \int_{\Gamma\backslash G} g(y)K(x,y)\,dy.$$

- Compute $\mathrm{Tr}\, I$ in 2 ways, geometrically + spectrally

$$\mathrm{Tr}\, I = \int_{\Gamma\backslash G} K(x,x)\,dx = \sum_{n\in\mathbb{Z}} f(n).$$

For $SL_2(\mathbb{Z})\backslash SL_2(\mathbb{R})$ = $A\log Y + T_2 + o(1)$
$\mathrm{Tr}\, K^Y = A\log Y + T_1 + o(1)$.

- For spectral expansion: Observe: $\Delta = \partial_{xx}$ Laplacian.

$$\mathcal{H} = L^2(\Gamma\backslash G),\quad \langle v,w\rangle = \int_{\Gamma\backslash G} v(x)\overline{w(x)}\,dx.$$

Δ is self-adjoint $\langle \Delta v, w \rangle = \langle v, \Delta w \rangle$.

$$\langle \partial_{xx} v, w \rangle \overset{\text{partial integration}}{=} -\langle \partial_x v, \partial_x w \rangle \overset{\text{again}}{=} \langle v, \partial_{xx} w \rangle.$$

$\tilde{I}$ is self-adjoint:

$$\langle \tilde{I} v, w \rangle \qquad \langle v, \tilde{I} w \rangle.$$

$$= \int_{T^G} \left(\int_{T^G} v(y) \overline{K(x,y)} dy \right) \overline{w(x)} \, dx$$

Also (semi-)definite operator:
If $\Delta v = \lambda v$. $\Rightarrow$
$\langle \Delta v, v \rangle = \lambda \langle v, v \rangle = -\langle \partial_x v, \partial_x v \rangle \leq 0.$

Know all eigenvalues
$\mathcal{H} = \bigoplus \mathbb{C} e_m$, $e_m(x) = e(mx)$,
$\Delta e_m = -4\pi^2 m^2 \leq 0$.

Claim: Δ & $\tilde{I}$ commute. (∂_x & $\tilde{I}$ commute).

pf: $$(\partial_x (\tilde{I} g))(x) = \lim_{h \to 0} \frac{1}{h} \left[(\tilde{I} g)(x+h) - \tilde{I} g(x) \right] = \lim \frac{1}{h} \int_{T^G} g(y) K(\cancel{x+h}, y) dy - \tilde{I} g(x)$$

$y \mapsto y+h$

$$= \int_{T^G} \left(\lim \frac{1}{h} \left[g(y+h) - g(y) \right] \right) K(x,y) dy = (\tilde{I} \partial_x g)(x).$$

Since Δ has orthonormal basis, $\tilde{I}$ is also orthonormalized by same. $\Delta e_m = \underbrace{\lambda_m}_{-4\pi^2 m^2} e_m$.

$$\tilde{I} e_m = \underline{\mu_m} e_m \quad ?$$

$$\underline{\mu_m = \hat{f}(m).}$$

Trick: No longer need to know that $e_m \in L^2(\mathbb{R})$ can take **any** eigenfunction of Δ on G with same eigenvalue to compute μ_m.

$$\hookrightarrow = \int_{x+G} e(m \cdot y) K(x,y)\, dy = \int_G e(my) k(x,y)\, dy = \hat{f}(m) \underbrace{e(mx)}_{= e_m}.$$

$$\Rightarrow \sum f(n) = \sum \hat{f}(m). \leftarrow \text{sum of eigenvalues of } T.$$

Riemann's Memoir via Tate.

Need: Mellin transforms $f : \mathbb{R}^\times_{>0} \to \mathbb{C}$. ($\mathbb{R}^\times_{>0} = G$)

$$\tilde{f}(\underset{\in \mathbb{C}}{s}) = \int_0^\infty f(y) \underbrace{y^s}_{\text{Character.}} \underbrace{\frac{dy}{y}}_{\text{Haar measure on } G}$$

$(xy)^s = x^s \cdot y^s.$

Eg: $f(y) = e^{-y}$ $(\operatorname{Re} s > 0)$ $\tilde{f}(s) = \int_0^\infty e^{-y} y^s \frac{dy}{y} = \Gamma(s).$

Thm (Mellin inversion): f "nice", $\frac{1}{2\pi i} \int_{(2)} \tilde{f}(s) y^{-s}\, ds = f(y)$

$(2) \leftarrow \operatorname{Re} s = 2.$

Pf 1: This is Fourier inversion (Exercise).

Hint: $\tilde{f}(s) = \int_0^\infty f(y) y^s \frac{dy}{y}$, $y = e^u$ $\longrightarrow = \int_{-\infty}^\infty f(e^u) e^{us}\, du.$

pf 2: (Goldfeld-K)

$$\boxed{dy = e^u\,du, \quad \frac{dy}{y} = du}$$

Why "should" Mellin inversion be true?

"δ-spike at $u=y$"

$$\frac{1}{2\pi i}\int_{(2)} \left[\int_0^\infty f(u)\, u^s \frac{du}{u}\right] y^{-s}\, ds = \int_0^\infty f(u) \left[\frac{1}{2\pi i}\int_{(2)} \left(\frac{u}{y}\right)^s ds\right]\frac{du}{u}.$$

Idea: Use partial integration for more smoothness:

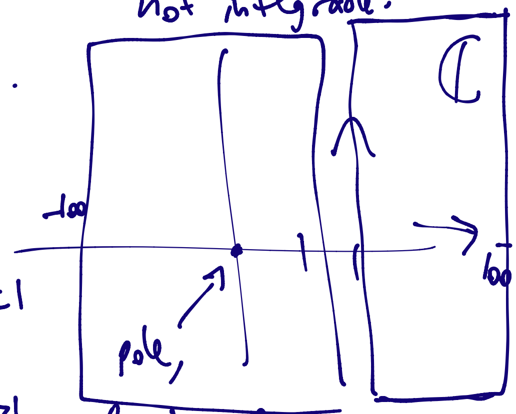
IE: $$\tilde f(s) = \int_0^\infty f(u)\, \underbrace{u^s \frac{du}{u}}_{u^{s-1}\,du} = -\int_0^\infty f'(u)\,\frac{u^s}{s}\, du,$$

So: $$\frac{1}{2\pi i}\int_{(2)}\left[-\int_0^\infty f'(u)\,\frac{u^s}{s}\,du\right] y^{-s}\, ds$$

$$= -\int_0^\infty f'(u)\left[\frac{1}{2\pi i}\int_{(2)}\left(\frac{u}{y}\right)^s\frac{ds}{s}\right]du.$$

Can't reverse orders, not integrable.

Look at: $$\frac{1}{2\pi i}\int_{(2)} x^s\, \frac{ds}{s} = \begin{cases} 0, & x<1 \\ 1, & x>1 \end{cases}$$



If $X<1$

(Notice: if $f(u) = \mathbb{1}_{u>1}$, then $\tilde f(s) = \int_1^\infty 1\cdot u^s\,\frac{du}{u} = \frac{1}{s}$.

$\frac{1}{2\pi i}\int_{(2)} \tilde f(s)\, x^{-s}\, ds$

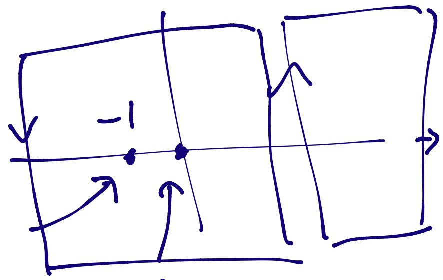
$$= -\int_0^\infty f'(u)\,\mathbb{1}_{\frac{u}{y}>1}\, du = -\int_y^\infty f'(u)\,du = f(y) \quad \text{FTC}.$$

"Real" pf: $\hat{f}(s) = -\int_0^\infty f'(y)\, \frac{y^s}{s}\, dy \overset{\text{again}}{=} \int_0^\infty f''(y)\, \frac{y^{s+1}}{\underset{\text{integrable!}}{s(s+1)}}\, dy.$

Look at: $\frac{1}{2\pi i}\int_{(2)} \left[\int_0^\infty f''(u)\, u \cdot \frac{u^s}{s(s+1)}\, du\right] y^{-s}\, ds = \int_0^\infty f''(u)\cdot u \left[\frac{1}{2\pi i}\int_{(2)} \frac{\left(\frac{u}{y}\right)^s}{s(s+1)}\, ds\right] du.$

What is $\frac{1}{2\pi i}\int_{(2)} \frac{x^s}{s(s+1)}\, ds = \begin{cases} 0, & x<1 \\ 1-\frac{1}{x}, & x>1 \end{cases}$

(Perron integral)



$= \int_0^\infty f''(u)\cdot u\, \mathbb{1}_{u>y} \cdot \left(1 - \frac{y}{u}\right) du.$

$= \int_y^\infty f''(u)\cdot(u-y)\, du = -\int_y^\infty f'(u)\cdot 1 \cdot du = f(y).$

To setup Riemann's memoir, play with Poisson Summation.

If $f_t(x) := f(xt) = (\pi(t).f)(x)$, $t \in G = \mathbb{R}^\times_{>0}$.

Exercise: $\hat{f}_t(\xi) = \frac{1}{t}\hat{f}\left(\frac{\xi}{t}\right).$

So Poisson Summation: $\sum_{n\in\mathbb{Z}} f_t(n) = \sum_{m\in\mathbb{Z}} \hat{f}_t(m)$

$$\boxed{\sum_{n\in\mathbb{Z}} f(nt) = \frac{1}{t}\sum_{m\in\mathbb{Z}} \hat{f}\left(\frac{m}{t}\right)}$$

Let $f(x) = e^{-\pi x^2}$ Gaussian.

Then Exercise: $\hat{f}(\xi) = e^{-\pi\xi^2}$.

Poisson $\Longrightarrow$ $\displaystyle\sum_{n\in\mathbb{Z}} e^{-\pi n^2 t^2} = \frac{1}{t}\sum_{m\in\mathbb{Z}} e^{-\pi m^2/t^2}$.

$t = \frac{1}{100}$.

no contribution once $\pi n^2 t^2 > 10$.

$\equiv |n| \gtrsim 100$.

once you sum 100 terms, stop.

$e^{-\pi 10000 m^2}$

Only term here is $m=0$.

$= \underline{100}$.