Last time: $\Gamma_{\mathcal{A}} = \left\langle \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} ; a \in \mathcal{A} \right\rangle^{+} \cap SL_2.$

$$R_N(n) = \sum_{\gamma \in \Gamma_{\mathcal{A}} \cap B_N} \mathbb{1}_{n = \langle v, \gamma v \rangle} = \int_0^1 \hat{R}_N(\theta) e(-n\theta)\, d\theta \qquad \boxed{v = e_1 = (1,0)^t}$$

$$= \int_{\mathfrak{M}} + \int_{\mathfrak{m}} = \mathcal{M}_N(n) + \mathcal{E}_N(n)$$

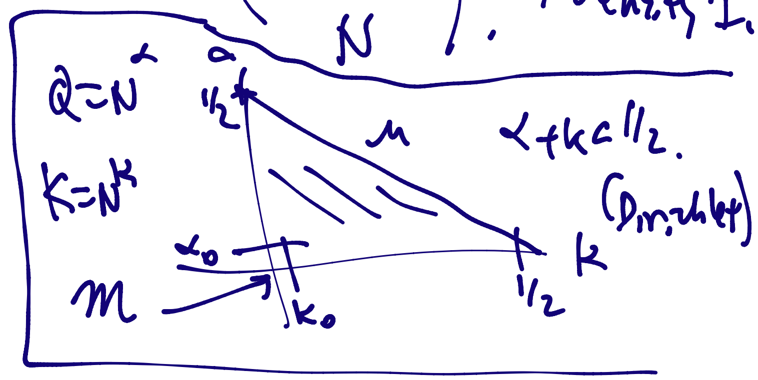
$$\hat{R}_N(\theta) = \sum_{\gamma \in \Gamma_{\mathcal{A}} \cap B_N} e(\theta \langle v, \gamma v \rangle).$$

$\hat{R}_N(0) = \# \Gamma_{\mathcal{A}} \cap B_N \asymp N^{2\delta}$

$$W_{Q,K} = \left\{ \theta = \frac{a}{q} + \beta : q \asymp Q, (a,q) = 1, |\beta| \asymp \frac{K}{N} \right\}$$

Need: $\sum_n |\mathcal{E}_N(n)|^2 = \int_{\mathfrak{m}} |\hat{R}_N(\theta)|^2 d\theta = o\left(\frac{N^{4\delta}}{N} \right)$. $\Rightarrow$ density 1.

$$= \sum_{\substack{Q < N^{1/2} \\ > \mathfrak{m} \\ \text{dyadic}}} \sum_{\substack{K < N^{1/2}/Q \\ \text{dyadic} \\ \alpha > \alpha_0 \text{ or} \\ K > K_0}} \int_{W_{Q,K}} |\hat{R}_N(\theta)|^2 d\theta$$



$$|W_{Q,K}| \approx Q \cdot Q \cdot \frac{K}{N}$$

To estimate in L^∞, need

For $\theta \in W_{Q,K}$, $|\hat{R}_N(\theta)| = o\left(\frac{N^{2\delta}}{Q \cdot K^{1/2}} \right)$

$$\hat{R}_N(\theta) = \sum_{\gamma \in \Gamma \cap B_N} e(\theta \langle v, \gamma v \rangle)$$

$\theta \downarrow \frac{a}{q} + \beta$

Want to apply harm. analysis in Γ. Can do this effectively when q & β are small (Major arcs).

New Idea: Steal idea from Vinogradov: create bilinear forms.

Replace $R_N(n)$ by: $\hat{R}_N(\theta) = \sum_{\gamma_1 \in \Gamma \cap B_X} 1 \cdot \sum_{\gamma_2 \in \Gamma \cap B_Y} e(\theta \langle v, \gamma_1 \gamma_2 v \rangle)$

$\boxed{X \cdot Y = N}$ (maybe $X = Y = N^{1/2}$).

Still have: $R_N(n) \neq 0 \Rightarrow n \in D_{\mathcal{A}}$.

In $\mathbb{Z} \cap B_X + \mathbb{Z} \cap B_Y \supset \mathbb{Z} \cap B_{X+Y}$. For $X + Y = N$, choose $X, Y = N/2$.

Cauchy-Schwarz:

$y \in \mathbb{Z}^2 \cap B_X$ $\langle \gamma_1 v, \gamma_2 v \rangle$

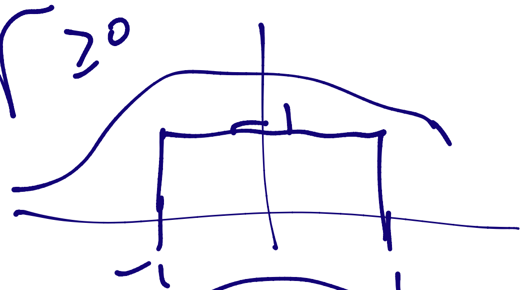
$$|\hat{R}_N(\theta)|^2 \le \left(\sum_{\gamma_1 \in \Gamma \cap B_X} 1^2 \right) \cdot \left(\sum_{\gamma_1 \in \Gamma \cap B_X} \left| \sum_{\gamma_2 \in \Gamma \cap B_Y} e(\theta \langle v, \gamma_1 \gamma_2 v \rangle) \right|^2 \right)$$

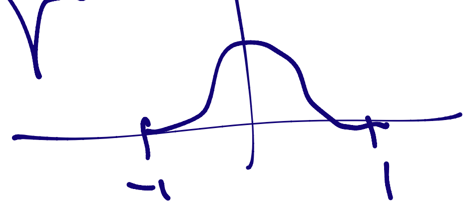
$$\lesssim X^{2\delta} \sum_{y \in \mathbb{Z}^2} \Upsilon\left(\frac{y}{X}\right) \left| \sum_{\gamma_2 \in \Gamma \cap B_Y} e(\theta \langle y, \gamma_2 v \rangle) \right|^2 .$$

Upsilon

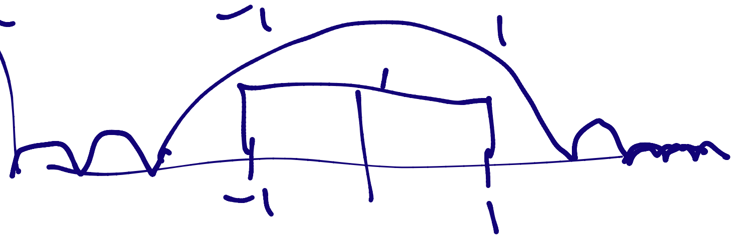
gain: harmonic analysis
lose: X^2 vs $X^{2\delta}$.

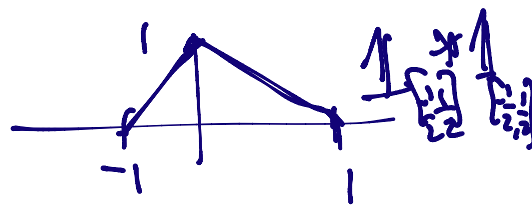
where $\Upsilon \ge 0$, $\operatorname{supp} \hat{\Upsilon} \subset [-1, 1]$.





$\Upsilon = \left(\frac{\sin x}{x} \right)^2$





$$\leq X^{2\delta} \cdot \sum_{\gamma_2, \gamma_2' \in \Gamma \cap B_Y} \underbrace{\sum_{y \in \mathbb{Z}^2} \Upsilon\left(\frac{y}{X}\right) e(\theta \langle y, (\gamma_2 - \gamma_2') v \rangle)}$$

(After squaring, Need to save $Q^2 \cdot K + \dots$)

Poisson summation

$$= \sum_{k \in \mathbb{Z}^2} \underbrace{\int_{y \in \mathbb{R}^2} \Upsilon\left(\frac{y}{X}\right) e(\theta \langle y, (\gamma_2 - \gamma_2') v \rangle) \, e(-k \cdot y) \, dy}_{y \mapsto y \cdot X}.$$

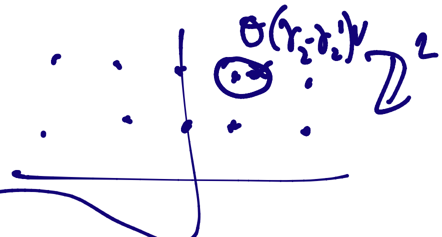
$$X^2 \underbrace{\int_{y \in \mathbb{R}^2} \Upsilon(y) \, e((-kX + \theta X (\gamma_2 - \gamma_2') v) \cdot y) \, dy}_{\hat{\Upsilon}(X(k - \theta(\gamma_2 - \gamma_2')v))}.$$

$$\Rightarrow |\hat{R}_N(\theta)|^2 \leq X^{2\delta} \overbrace{\sum_{\gamma_2, \gamma_2' \in \Gamma \cap B_Y}}^{X^2} \cdot \sum_{k \in \mathbb{Z}^2} \hat{\Upsilon}(X(\underbrace{k - \theta(\gamma_2 - \gamma_2')v}))$$

$$\boxed{\text{this loss is only acceptable if } \delta \text{ "near" } 1.}$$

$\neq 0$ only when $|X(k - \theta(\gamma_2 - \gamma_2')v)| < 1$

i.e. only when $|\theta(\gamma_2 - \gamma_2')v - k| < \frac{1}{X}$.



At most 1 value of $k \in \mathbb{Z}^2$ is close enough, need $\|\theta(\gamma_2 - \gamma_2')v\| < \frac{1}{X}$.

distance to $\mathbb{Z}^2$.

$$\Rightarrow |\hat{R}_N(\theta)|^2 \leq X^{4\delta} X^{2(1-\delta)} \sum_{\gamma_2, \gamma_2' \in \Gamma \cap B_Y} \mathbb{1}_{\|\theta(\gamma_2 - \gamma_2')v\| < \frac{1}{X}}.$$

$$\left\|\frac{a}{q}(\gamma_2-\gamma_2')v\right\| \leq \|\theta(\gamma_2-\gamma_2')v\| + |\beta(\gamma_2-\gamma_2')v|$$

(in $\theta \in W_{Q,K}$)

$$< \frac{1}{X} + \frac{K}{N}\cdot Y$$

$\boxed{\text{Choose } X = Y = N^{1/2}}$

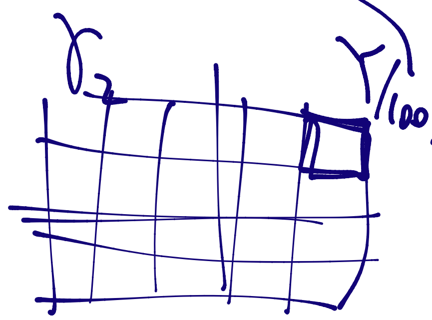
$$< \frac{1}{N^{1/2}} + \frac{K}{N^{1/2}} < \frac{K}{N^{1/2}} < \frac{1}{Q}.$$

If $\gamma_2 v \not\equiv \gamma_2' v \bmod q$.

then $\geq \frac{1}{q} > \frac{1}{Q}$

But $Q \cdot K < N^{1/2}$.

So $\|\theta(\gamma_2-\gamma_2')v\| < \frac{1}{X} \Rightarrow \gamma_2 v \equiv \gamma_2' v \ (q)$.



$$\Rightarrow |\hat{R}_N(\theta)|^2 \ll X^{4\delta} X^{2(1-\delta)} \cdot \sum_{\gamma_2 \in \Gamma \cap B_Y} \cdot \sum_{\gamma_2' \in \Gamma \cap B_Y} \mathbb{1}\left\{\begin{array}{l}\gamma_2' v \equiv \gamma_2 v \ (q) \\ |\gamma_2' v - \gamma_2 v| < \frac{1}{X|\beta|}\end{array}\right.$$

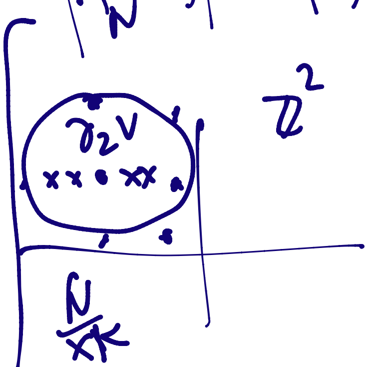
Once $\gamma_2 v \equiv \gamma_2' v \pmod q \Rightarrow$

$\frac{1}{X} > \|\theta(\gamma_2-\gamma_2')v\| = \|\beta(\gamma_2-\gamma_2')v\| = |\beta(\gamma_2-\gamma_2')v|$

$\gamma_2' v = z \in \mathbb{Z}^2$.

$\boxed{\text{Again losing } Y^{2\delta} \to Y^2}$

$$\Rightarrow |\hat{R}_N(\theta)|^2 \ll X^{4\delta} X^{2(1-\delta)} \sum_{\gamma_2 \in \Gamma \cap B_Y} \cdot \boxed{\sum_{z \in \mathbb{Z}^2 \cap B_Y} \mathbb{1}\left\{\begin{array}{l} z \equiv \gamma_2 v \pmod q \\ |z - \gamma_2 v| < \frac{N}{XK}\end{array}\right\}}.$$



$$\hookrightarrow \ll \frac{N^2}{X^2 K^2 Q^2} \cancel{+ K}$$

$QK < N^{1/2}$.

$X = Y = N^{1/2}$

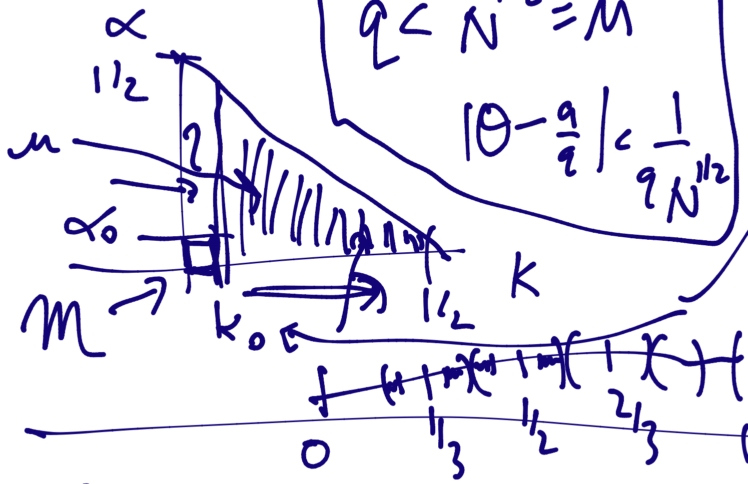
$$\frac{N}{K^2 Q^2} \geq 1$$

$$|\hat{R}_N(\theta)|^2 \ll X^{4\delta} X^{2(1-\delta)} \cdot Y^{2\delta} \frac{X \cdot Y}{K^2 Q^2} = \frac{N^{4\delta} N^{2(1-\delta)}}{Q^2 \cdot K \cdot K}.$$

So: as long as $\boxed{K > N^{2(1-\delta)}}$, this wins.

$$\boxed{\begin{array}{l} q < N^{1/2} = M \\ |\theta - \frac{a}{q}| < \frac{1}{q N^{1/2}} \end{array}}$$

If κ_0 is large ~~small~~ enough that $\kappa > 2(1-\delta)$,



Major arcs

$q < Q_0 = N^{\alpha_0}$, $|\beta| < \frac{K_0}{N}$

$K_0 = N^{\kappa_0}$.

Remains: to estimate $\int_{W_{Q,K}} |\hat{R}_N(\theta)|^2 d\theta$ when K small, $K = N^{\kappa}$, $\kappa < \kappa_0$ but Q is large, $Q = N^{\alpha}$, $\underline{\alpha > \alpha_0}$.

Let $P_{Q,\beta} = \left\{ \theta = \frac{a}{q} + \beta \;\middle|\; q \asymp Q,\ (a,q) = 1 \right\}$.

$|P_{Q,\beta}| \approx Q^2$.

Kloosterman-type analysis ("refinement"): want to try to get cancellation in $\sum_{\theta \in P_{Q,\beta}} |R_N(\theta)|$.

Foreshadowing: $\displaystyle\sum_{\theta\in P_{Q,\beta}} \cdot \underbrace{\int_\theta}_{|\cdot|=1} \cdot \underbrace{\sum_{\gamma_1\in\Gamma\cap B_x}}\ \sum_{\gamma_2\in\Gamma\cap B_y} e(\theta\langle v, \gamma_1\gamma_2 v\rangle)$.

pull γ_1 out,

C-S in γ_1, want cancellation in $\underbrace{\gamma_2', \underbrace{q, q', a, a'}}$.

Exercise: Try to work this out yourself without

$$\sum_{\theta\in P_{q,\beta}} |\hat{R}_N(\theta)|, \quad \text{where } P_{q,\beta} = \left\{ \frac{a}{q}+\beta \;\middle|\; \underline{(a,q)=1} \right\}.$$

L^∞

$$\int_{W_{Q,K}} |\hat{R}_N(\theta)|^2 d\theta = \int_{|\beta|\lesssim \frac{K}{N}} \left(\sum_{\theta\in P_{Q,\beta}}' |\hat{R}_N(\theta)| \cdot |\hat{R}_N(\theta)| \right) d\beta$$