Last time: inverse map $a \mapsto \bar{a} \ (p)$, random,

Kloosterman sums: $S_p(n,m) = \sum'_{a(p)} e_p(an + \bar{a}m)$

$\ll p^{3/4}$. $((n,m) \neq (0,0))$!

multiplication: $a \mapsto a \cdot b \mod p$.

Saw: "quality" of randomness depended on

$\frac{b}{p} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cdots}} = [0; a_1, a_2, \ldots, a_\ell]$, $A = \max a_j$.

Thm Zaremba '69: $\mathrm{Disc}\left(\left\{\left(\frac{a}{p}, \frac{b \cdot a}{p}\right) \bmod 1 : a \in \mathbb{Z}/p\right\}\right)$

$\ll A \cdot \frac{\log p}{p}$ (p arbitrary)

Recall Schmidt $\frac{\log p}{p}$.

But $A = A\left(\frac{b}{p}\right)$ is not a constant!

Conj Zaremba '72: $\exists A < \infty$ s.t. $\forall d \geq 1$, $\exists (b,d)=1$: $\frac{b}{d} = [0; a_1, \ldots, a_\ell]$, & all $a_j \leq A$.

Conj: $A = 5$. $A \neq 4$. Look at $d = 6$.

b: 1, $\frac{1}{6} = [0; 6]$ - too big.

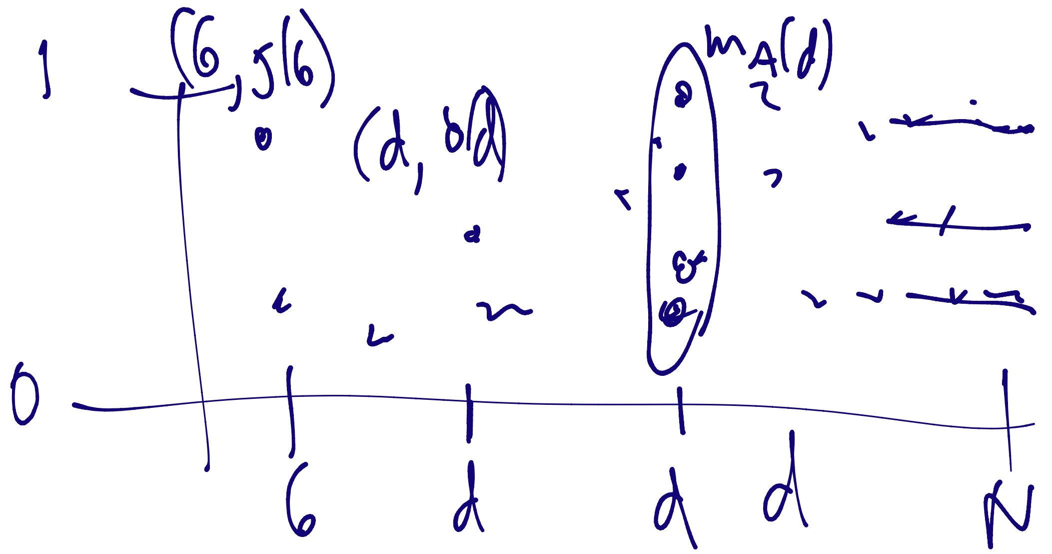
5, $\frac{5}{6} = [0; 1, 5]$.

$\boxed{6 \notin D_4}$.

Let $R_A = \left\{ \frac{b}{d} = [0; a_1, \ldots, a_\ell] \mid a_j \leq A \right\}$. $R_A \subseteq R_{A+1}$

Let $D_A = \left\{ d \geq 1 : \exists (b,d)=1, \frac{b}{d} \in R_A \right\}$.

Conj: $D_5 = \mathbb{N}_{\geq 1}$. Look at R_A graded by d



$A=5$

$\left(d, \frac{b}{d}\right)$.
$\Uparrow$
R_A.

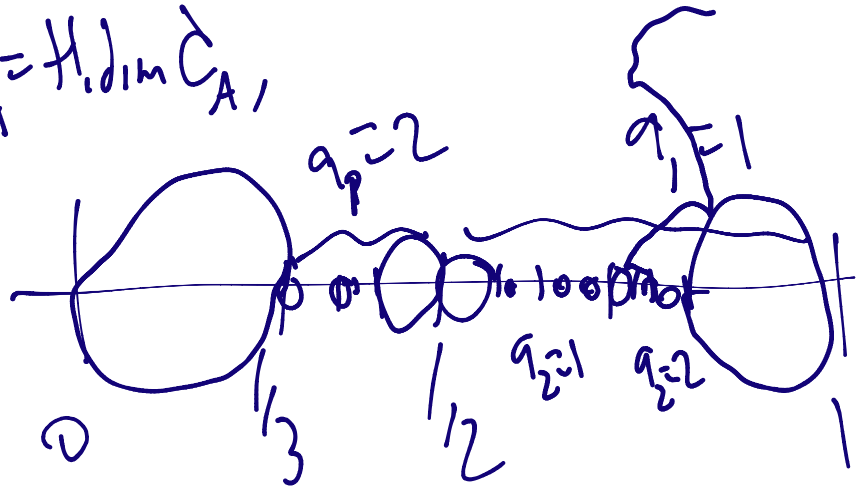
Let $\Lambda(R_A) = \underline{\mathcal{C}_A}$ = limit pts of R_A.

$$\boxed{x \mapsto 1/x - \underbrace{\lfloor 1/x \rfloor}_{a_j}} = \Big\{ [0; \overset{1 \text{ or } 2}{a_1}, \overset{1 \text{ or } 2}{a_2} \dots] \;\Big|\; \text{all } a_j \le A \Big\}.$$

Eg $A=2$, $\delta_A = \text{H.dim}\, \mathcal{C}_A$, H

(1940s) $\mathcal{C}_2$:

$\delta_2 = 0.531\ldots$



Let $R_A(N) := \left\{ \frac{b}{d} \in R_A, \; (b,d)=1, \; d < N \right\}$.

Q: $\# R_A(N)$? Trivial $\# R_A(N) \ll N^2$.

To study this, consider $\mathcal{L}_A(s) := \sum_{d \ge 1} \frac{m_A(d)}{d^s}$.

$m_A(d)$ = multiplicity $= \#\{(b,d)=1,\ \frac{b}{d} \in R_A\}$.

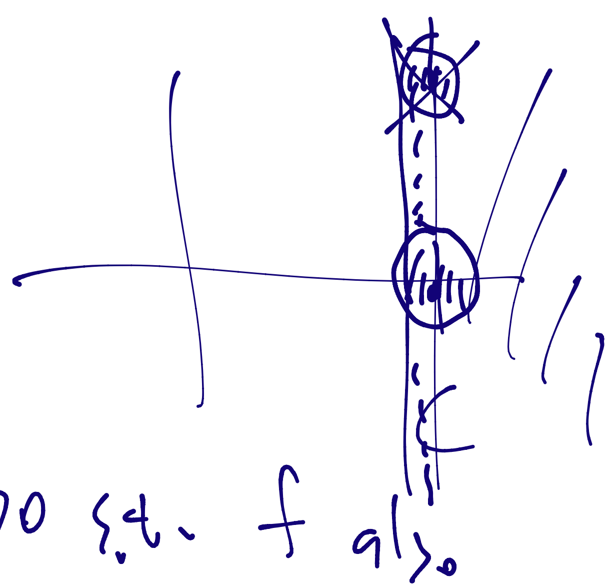
Then $\sum_{d \le N} m_A(d) = \# R_A(N)$.

<u>Landau's Lemma:</u> Given Dirichlet series

$f(s) = \sum_{n \ge 1} \frac{a_n}{n^s}$, $\underline{\underline{a_n \ge 0}}$. Converges abs in

some $\mathrm{Re}(s) > C$. If

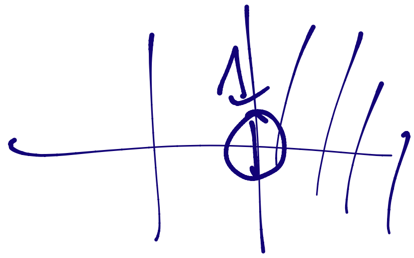
it has hol. cont to



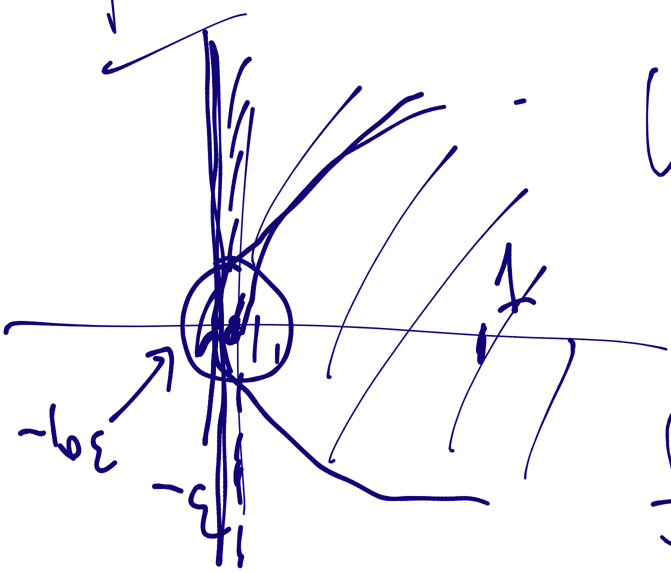
<u>neigh of C</u>, Then $\exists\, \varepsilon > 0$ s.t. f abs

Converges abs for $\mathrm{Re}(s) > C-\varepsilon$.

Absolutely false for Dirichlet L-functions!

$L(\chi, s)$ entire 

Pf: Exercise! Can Assume $C=0$.



Look at Taylor series expansion of f around 1.

$$f(s) = \sum_{k \geq 0} \frac{f^{(k)}(1)}{k!} (s-1)^k .$$

Converges abs for $|s-1| < 1+\varepsilon$. for some $\varepsilon > 0$.

$$f^{(k)}(s)\Big|_{s=1} = \sum a_n e^{-s \log n} (-\log n)^k \Big|_{s=1}$$

$$= \sum_{n\geq 1} \frac{a_n (-\log n)^k}{n}.$$

$f(-\varepsilon)$ $\underline{\text{conv abs.}}$

$$= \sum_{k\geq 0} \frac{1}{k!} (1+\varepsilon)^k \cancel{(-1)}^k \left\| \left[\sum_{n\geq 1} \frac{a_n (\log n)^k}{n} \right] \right\|$$

$$= \sum_{n\geq 1} \frac{a_n}{n} \underbrace{\sum_{k\geq 1} \frac{1}{k!}\left[(1+\varepsilon)\log n\right]^k}_{\exp((1+\varepsilon)\log n) = n^{1+\varepsilon}}.$$

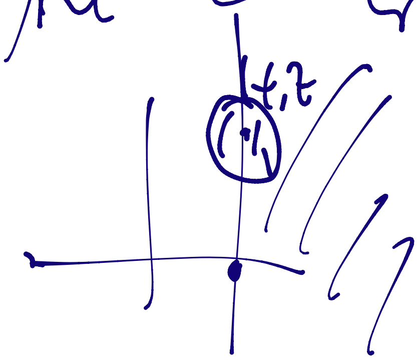
$$\infty > = \sum_{n\geq 1} \frac{a_n}{n} \cdot n^{1+\varepsilon} = \sum_{n\geq 1} \frac{a_n}{n^{-\varepsilon}} = \sum \left| \frac{a_n}{n^{-\varepsilon + it}} \right|$$

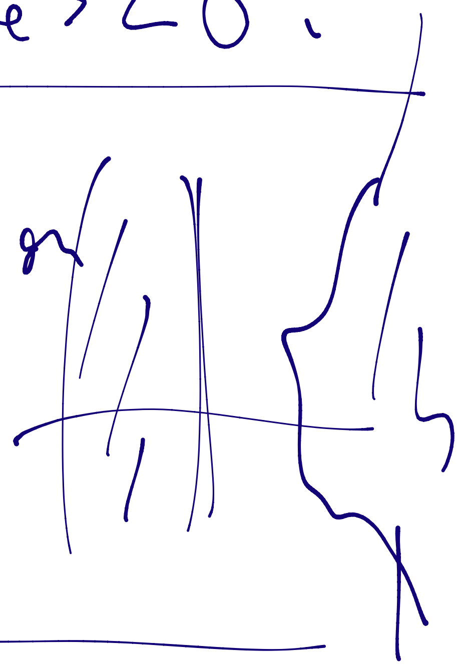
$\Rightarrow f$ $\underline{\text{conv abs}}$ on $\mathrm{Re}\, s > -\varepsilon$.

Cor: If f is a Dirichlet series with non-neg coefficients, then it has an abscissa of convergence, i.e. $\exists \sigma$ s.t. f conv abs on $\mathrm{Re}(s) > \sigma$ & not on $\mathrm{Re}\, s < \sigma$.

Might imagine converges on

$\zeta(1+it+s)$





Back to $\mathcal{L}_A(s) = \sum_{d \geq 1} \frac{m_A(d)}{d^s}$ ← conv abs $\mathrm{Re}(s) > 2$.

$\# R_A(x) = \sum_{d \leq N} m_A(d) \ll N^2$

Has some abs. of conv. at $\sigma_A = 2\delta_A$.

Aside: $\mathfrak{C} = \bigcup_{A<\infty} \mathfrak{C}_A$ = Bad'y approx numbers.

$m(\mathfrak{C}) = 0$, $\mathrm{Hdim}\,\mathfrak{C} = 1. \leftarrow \delta_A$ as $A \to \infty$.

$$\#R_A(N) = \frac{1}{2\pi i}\int_{(2)} \mathcal{L}_A(s) N^s ds$$

pull contour to $\mathrm{Re}(s) = 2\delta_A + \varepsilon$.

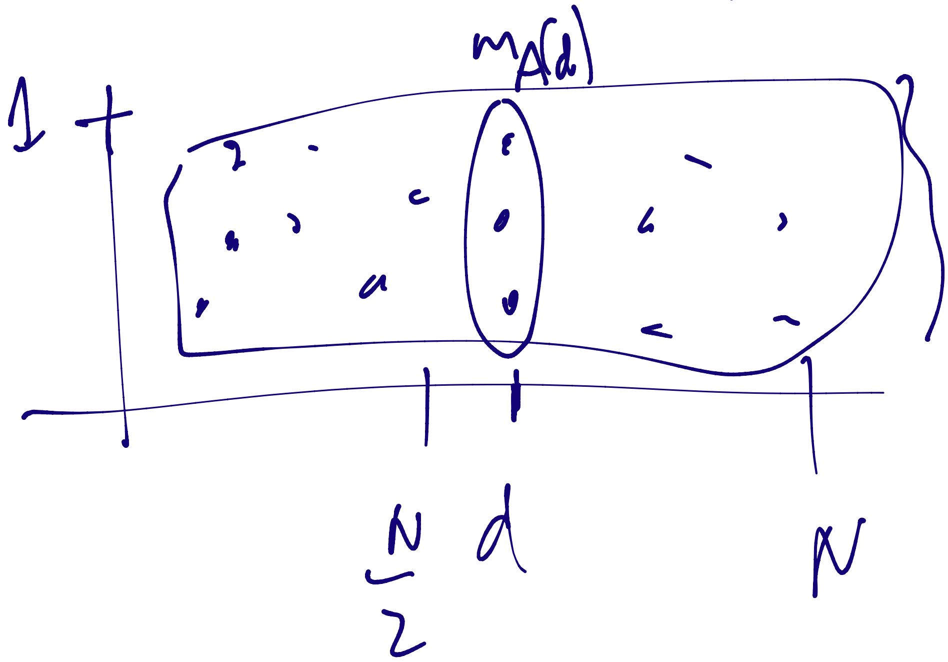
$$\Rightarrow \quad \#R_A(N) \ll N^{2\delta_A+\varepsilon}$$

OTOH If $\#R_A(N) \ll N^{2\delta_A - \varepsilon} \Rightarrow \mathcal{L}_A(s)$ would converge absolutely

$\Rightarrow \quad \# R_A(N) = N^{2\delta_A + o(1)}$ ← Hensley '89

← Lalley '89

What might we expect for $m_A(d)$?



$$\begin{array}{ccc} R_A & \ni & \frac{b}{d} \\ \downarrow & & \downarrow \\ D_A & \ni & d \end{array}$$

What happens? Universe 1 very few d's hit very often. Universe 2: d's $\lesssim N$ hit roughly equally. $\Rightarrow m_A(d) \approx \frac{N^{2\delta_A}}{N}$.

$\delta_2 = 0{,}531\ldots, \Rightarrow 2\delta_2 - 1 > 0.$ $\qquad = N^{2\delta_A - 1}.$

$\Rightarrow$ for N (& $d \sim N$) large$^{\to\infty}$, $0 < m_A(d) \to \infty$;

Hensley Gary '92: $\mathcal{D}_2 \supseteq \mathbb{N}_{\gg 1}$.

Thm (Bourgain-K '14): $\exists A < \infty$ ($A = 50$, $= 5$),

s.t. 100% of numbers are in $\mathcal{D}_A$,

$$\frac{1}{N} \# \mathcal{D}_A \cap [1, N] \to 1.$$

$$\# \mathcal{D}_A \cap [1, N] = N + o(N)$$

Conj: $\# \mathcal{D}_A \cap [1, N] = N + O(1)$.
