Last time: $E(z,s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} (\mathrm{Im}\, \gamma z)^s$. Fourier expansion

$z+1$

$$\underbrace{\pi^{-s}\Gamma(s)\zeta(2s)}_{\xi(2s)} E(z,s) = 2\left[\xi(2s)y^s + \xi(2s-1)y^{1-s}\right] + \left(2 \cdot K_{s-\frac{1}{2}}(\pi|k|y)\right)$$

$$2\sum_{k\neq 0} y^{1/2} |k|^{s-1/2} \sigma_{1-2s}(|k|) \int_0^\infty e^{-\pi|k|y(u+\frac{1}{u})} u^{s-\frac{1}{2}} \frac{du}{u}$$

"Nature" of this integral — "K-Bessel".

Mellin transform: $f : \mathbb{R}_{>0} \to \mathbb{C}$, $\tilde{f}(s) = \int_0^\infty f(y) y^s \frac{dy}{y}$. ($s \in \mathbb{C}$)

$$\hat{\tilde{f}}(y) = \frac{1}{2\pi i}\int_{(2)} \tilde{f}(s) y^{-s} ds = f(y).$$

$(2) \to 2 - i\infty \to 2 + i\infty$.

Mellin Convolution: $f * g(y) = \int_0^\infty f(u) g\left(\frac{y}{u}\right) \frac{du}{u}$.

Exercise: $\widetilde{f*g} = \tilde{f}\cdot\tilde{g}$

$$(\tilde{f} * \tilde{g})(s) = \frac{1}{2\pi i}\int_{(2)} \tilde{f}(w)\tilde{g}(s-w)\, dw$$

$\widehat{\tilde{f} * \tilde{g}} = \hat{\tilde{f}} \cdot \hat{\tilde{g}}$
$= f \cdot g.$

Look at $f_w(y) = e^{-y^2} y^w$

$u = y^2$, $\frac{du}{u} = \frac{2y\,dy}{y^2}$

$$\tilde{f}_w(s) = \int_0^\infty e^{-y^2} y^w y^s \frac{dy}{y} = \frac{1}{2}\int_0^\infty e^{-u} u^{\frac{s+w}{2}} \frac{du}{u} = \frac{1}{2}\Gamma\left(\frac{s+w}{2}\right).$$

Compute $(f_w * f_{-w})(y) = \int_0^\infty e^{-u^2} u^w e^{-(y/u)^2} \left(\frac{y}{u}\right)^{-w} \frac{du}{u}$

$e^{-(u^2+y^2/u^2)}$

$v = \frac{u^2}{y}, \quad \frac{dv}{v} = \frac{2u\,du \cdot y}{y\, u^2}$

$= \frac{1}{2}\int_0^\infty e^{-(yv+y/v)} (yv)^w \cancel{y}^{-w} \frac{dv}{v}$

"Bible" Gradshteyn-Ryzhik

$= \boxed{\frac{1}{2}\int_0^\infty e^{-y(v+1/v)} v^w \frac{dv}{v}} =: K_w(y)$. entire in w

Quiz: $\widetilde{K_w}(s) = \widetilde{f_w * f_{-w}}(s) \neq \widetilde{f_w}(s)\cdot \widetilde{f_{-w}}(s) = \frac{1}{4}\Gamma\left(\frac{s+w}{2}\right)\Gamma\left(\frac{s-w}{2}\right)$

Quiz: $\boxed{\frac{1}{2\pi i}\int_{(2)} \frac{1}{4}\Gamma\left(\frac{s+w}{2}\right)\Gamma\left(\frac{s-w}{2}\right) y^{-s}\, ds} = K_w(y)$

$\pi^{-s}\Gamma(s)\zeta(2s)E(z,s) \overset{\operatorname{Re} s>1}{=} 2\left[\zeta(2s)y^s + \overset{\zeta(2(1-s))}{\overset{\shortparallel}{\xi(2s-1)}} y^{1-s}\right] +$

$1-(2s-1) = 2(1-s)$

$\xi(2s) \leftarrow$ poles at $2s = 0, 1$

$4\sum_{k\neq 0} y^{1/2} |k|^{s-1/2} \sigma_{1-2s}(|k|) \, K_{s-\frac{1}{2}}(\pi|k|y)$ ← entire.

modular/arithmetic — archimedean

gives mer'c continuation of $E(z,s)$ to all of $\mathbb{C}$. $\operatorname{Re} s>1$

→ **Important:** (Langlands-Shahidi) Constant $\overset{(k=0)}{\text{terms}}$ of Eisenstein series give information on L-functions.

→ mer'c cont. poles at: $2s = 0,1$, $2(1-s) = 0,1$; $\Rightarrow$ $s=0$, $\cancel{s=\frac{1}{2}}$, $\cancel{s=\frac{1}{2}}$ exactly cancel, $s=1$

Obs: Constant term invariant under $s \to 1-s$.

modular part:

$\cancel{|k|^{s-1/2}} \sum_{ab=|k|} a^{s-1/2} b^{s-1/2} a^{1-2s} = \sum_{ab=|k|} \left(\frac{b}{a}\right)^{s-1/2}$

invariant under $s \to 1-s$!

$-(s-\tfrac{1}{2}) = (1-s) - \tfrac{1}{2}.$

$\boxed{=} \sum_{ab=|k|} \left(\frac{b}{a}\right)^{-(s-1/2)}.$

$K_w(y) = \frac{1}{2}\int_0^\infty e^{-y(u+1/u)} u^w \frac{du}{u} = \frac{1}{2}\int_0^\infty e^{-y(\frac{1}{u}+u)} u^{-w}\frac{du}{u}.$

$\boxed{u \mapsto 1/u}$

$= K_{-w}(y)$ even in "index".

$\shortparallel$

$(f_w * f_{-w})(y)$ even in w!

$K_{s-1/2}(y) = K_{(1-s)-1/2}(y).$

$\underbrace{(1-s)-1/2} = \tfrac{1}{2}-s = -(s-1/2).$

$y \geq 2$ & $\underbrace{u+1/u}_{\underbrace{2\cosh(\log u)}_{\geq 1.}} \geq 2 \Rightarrow y(u+1/u) \geq y + u + 1/u.$

$|K_w(y)| \leq \frac{1}{2}\int_0^\infty e^{-y} e^{-(u+1/u)} u^{\mathrm{Re}\, w}\frac{du}{u} \ll_w e^{-y}.$

So $K_{s-1/2}(\pi|k|y)$ once $|k| > \frac{2}{\pi y}$, $\ll e^{-\pi|k|y}.$

exp decay in $|k|$ in arch $\cdot$ poly growth in modular part

$\Rightarrow$ abs convergence of the series $\forall y$ $\boxed{\text{poles at } s=0,1.}$

$\Rightarrow$ mer'c cont of Eisenstein series, FE $s \to 1-s$.

$$\operatorname{Res}_{s=1} \underbrace{\zeta(2s)}_{\frac{\pi^2}{6}} E(z,s) = +2 \operatorname{Res}_{s=1} \zeta \quad , \quad \operatorname{Res}_{s=1} E(z,s) = \frac{3}{\pi} = \frac{1}{\mathrm{Vol}(\mathbb{H}/\Gamma)}$$

indep of z.

$$\zeta(2s) \sum_{[Q]\in \mathcal{C}\ell_D} \underline{E(\alpha_Q, s)} = \left(\frac{\sqrt{|D|}}{2}\right)^s \zeta_K(s) = \left(\frac{\sqrt{|D|}}{2}\right)^s \zeta(s)\, L(s,\chi)$$

mod $q \equiv 3(4)$ $\qquad D = -q$.

$\operatorname{Res}_{s=1}$ of both sides

$$\frac{\pi^2}{6} h(D) \cdot \frac{3}{\pi} = \frac{\sqrt{|D|}}{2} L(1,\chi) \qquad \boxed{x^2 - x - \frac{-q-1}{4}}$$

$\Rightarrow$ Dirichlet Class # Formula: $\boxed{q \equiv 3(4)}$

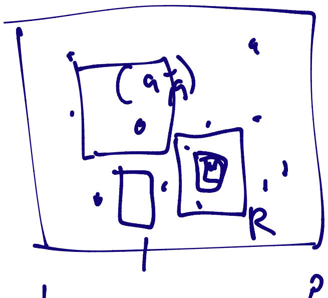
$$L(1,\chi) = \frac{h(-q)\cdot \pi}{\sqrt{q}} > 0.$$

≥ 1

$\Rightarrow$ Dirichlet's Thm $\displaystyle\sum_{p\equiv a(q)} \frac{1}{p} = \infty$. ✓

"Unrelated" problem: $p \left(= |D|\right)$ prime, $a \mapsto \bar{a}$ mod p. Is this "random"?

graph $(a, \bar{a})$ mod p.



How well can we guess what $\bar{a}$ is from a?

$$S_p \subset \mathbb{T}^2 = [0,1)\times[0,1) = (\mathbb{R}/\mathbb{Z})^2.$$

To measure this, $\mathrm{Discrepancy}\left(\overbrace{\left\{\left(\frac{a}{p}, \frac{\bar{a}}{p}\right) : a \in \mathbb{Z}_p^\times\right\}}^{S_p}\right)$

$$= \sup_{R\subset \mathbb{T}^2} \left| \frac{\# R \cap S_p}{\# S_p} - \mathrm{Area}(R) \right|$$

$\boxed{\begin{array}{l} \mathrm{Discr}(S) \geq \dfrac{1}{|S|} \ ✓ \\ \text{Schmidt: } \mathrm{Discr}(S) \gg \dfrac{\log |S|}{|S|} \end{array}}$

If $\mathrm{Discr} \to 0$ as $p \to \infty$ ✓, then points of S_p are becoming e.d.

Think of $f : \mathbb{T}^2 \to \mathbb{C}$ $\approx \mathbb{1}_R$. Then $\dfrac{\# R \cap S_p}{\# S_p} \approx \nu_p(f) = \langle f, \nu_p \rangle$,

$$\underbrace{\nu_p \text{ measure}} := \underbrace{\frac{1}{p-1} \sum_{a(p)}' \delta_{\left(\frac{a}{p}, \frac{\bar{a}}{p}\right)}}.$$

$$= \int f \, d\nu_p$$

$$= \frac{1}{p-1} \sum_{a(p)}' f\left(\frac{a}{p}, \frac{\bar{a}}{p}\right).$$

For e.d., need to show $\nu_p \xrightarrow{\text{weak-*}} \mu = \text{Haar on } \mathbb{T}^2$, Lebesgue.

i.e. $\forall$ fixed $f \in C^\infty(\mathbb{T}^2)$, $\nu_p(f) \to \mu(f) = \iint_{\mathbb{T}^2} f(x,y)\, dx\, dy$.

$$\langle \nu_p, f \rangle = \langle \hat{\nu}_p, \hat{f} \rangle$$

$$\hat{\nu}_p(n,m) = \langle \nu_p, e_p(n,m) \rangle = \frac{1}{p-1} \underbrace{\sum_{a(p)}' e_p\left(na + m\bar{a}\right)}_{\text{"Kloosterman" sum}}$$

$$= \frac{1}{p-1} \sum_{a(p)}' e^{\underbrace{\frac{2\pi i}{p}\left(na + \frac{m}{a}\right)}_{\text{modular "K-Bessel" function}}}.$$

In[∘]:=
```
p = Prime[10 003]
ListPlot[Table[{a, PowerMod[a, -1, p]}, {a, 1, p - 1}], AspectRatio → 1]
```

Out[∘]= 104 761

Out[∘]=