Last time: Riemannian Structure on $\mathbb{H}$, $T^1\mathbb{H}$, φ_t geodesic flow, $\Rightarrow$ Laplace-Beltrami for $\mathbb{H}$:

Fact: $\Delta = -y^2\left(\partial_{xx}+\partial_{yy}\right)$

Exercise: For $g \in G = (P)SL_2(\mathbb{R})$, let L_g be left operator, i.e. $(L_g f)(\overset{\in \mathbb{H}}{z}) = f(gz)$. Then

$$L_g \Delta = \Delta L_g.$$

proof 1: Direct tedious, uninspired computation.

pf 2: Lie algebra, $\mathfrak{g} = \{X \in M_{2\times 2}(\mathbb{R}) \mid \underbrace{\exp X}_{I+X+\frac{X^2}{2!}+\cdots} \in G\}$,

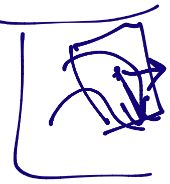
Claim: $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{g} \iff \operatorname{tr} X = 0.$

Exercise: $\det \exp X = \exp(\operatorname{tr} X)$.

Elements $\underbrace{X \in \mathfrak{g}}_{G.} \longrightarrow$ differential operators on $\underline{f : G \to \mathbb{C}}$,

$$\underbrace{(X.f)}_{:G\to\mathbb{C}}(\overset{G}{g}) = \frac{d}{dt} \underbrace{f(g.\overbrace{\exp(tX)}^{\sim I+tX \in G})}_{F_g(t)} \Big|_{t=0}.$$

$\boxed{ad-bc=1}$



Say $\underbrace{\mathfrak{g}}_{\text{vector space}} = \mathfrak{sl}_2(\mathbb{R})^{\subset \mathbb{R}^4} = \leftarrow$ basis, $X_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ (nilpotent), $X_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$,

$X_3' = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. $\underbrace{X_3 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}$:

$G = SL_2(\mathbb{R})$.

- $\exp(tX_1) = I + tX_1 = \overset{N}{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}}$, $(X_1^2 = 0)$.
- $\exp(tX_2) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$.
- $\exp(tX_3') = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$.
- $\exp(tX_3) = \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$ (?).

$$(\underset{\Uparrow \atop \mathfrak{g}}{X_3} . f)(\underbrace{n_x a_y k_\theta}) = \frac{d}{dt} f(n_x a_y \underbrace{k_\theta k_t})\Big|_{t=0}$$

$$= \frac{\partial}{\partial \theta} f(n_x a_y k_\theta).$$

$\mathbb{R} \times \mathbb{R} \times \mathbb{R}/2\pi \to G \xrightarrow{f} \mathbb{C}$

$(x, y, \underset{\Uparrow}{\theta}) \mapsto n_x a_y k_\theta$

$$(X_3 . f)(x, y, \theta) = \boxed{\frac{\partial}{\partial \theta}} f.$$

$\bigcup_k \overbrace{\mathfrak{g} \otimes \cdots \otimes \mathfrak{g}}^{k}$

$D \in U\mathfrak{g}$ Universal enveloping algebra = all orders of all derivatives

$X_3^2 \quad X_3(X_3 f) \quad Y_1, \dots, Y_k \in \mathfrak{g},$

$D = \underline{Y_k \cdots Y_2 Y_1}$.

Center of $U\mathfrak{g}$ = D's that commute with all others.

Center generated by C (2nd order operator).

on $C^\infty(G)$ Restrict C to Eigspace

$C^\infty(G/K) = C^\infty(G)^K$, in $n_x a_y k_\theta$ coordinates,

$K = SO(2)$

$C = \Delta$.

Why does Δ $\underline{\text{commute}}$ with L_g?

$(L_g f)(h) = f(\underbrace{g}_{\text{left}} h)$, $h \in G/K$. but $X \in \mathfrak{g}$,

$(X.f)(g) = \frac{d}{dt} f(g \underbrace{\exp(tX)}_{\text{right}}) \Big|_{t=0}$.

Easy to find e-funct of $\Delta = -y^2(\partial_{xx} + \partial_{yy})$ on $\mathbb{H}$,

$f(x+iy) = y^s = (\operatorname{Im} z)^s$. pf: $\partial_x f = 0$, $\partial_y f = s y^{s-1}$

$\partial_{yy} f = s(s-1) y^{s-2}$, $\Delta f = \underbrace{s(1-s)}_{\lambda} f$.

<u>Q:</u> Is $g(z) = \left(\frac{y}{(x+1)^2+y^2}\right)^s$ also e-funct of Δ?

<u>Mathematica! Yes.</u> $f \checkmark$, $g(z) = f\left(\underbrace{\begin{pmatrix}2&1\\1&1\end{pmatrix}}_{\in G} z\right)$.

$$\operatorname{Im}\left(\begin{pmatrix}2&1\\1&1\end{pmatrix} z\right)^s = \left(\frac{\operatorname{Im} z}{|z+1|^2}\right)^s = \left(\frac{y}{(x+1)^2+y^2}\right)^s.$$

Allows to construct Γ-invariant eigenfunctions of Δ:

"nonholomorphic" Eisenstein series. e-funct $\forall g \in G$.

$$E(z,s) = \sum_{\gamma \in T_\infty \backslash \Gamma} (\operatorname{Im} \gamma z)^s = E(\gamma z, s). \quad \forall \gamma \in \Gamma = SL_2(\mathbb{Z}).$$

$T_x = \begin{pmatrix}1&x\\0&1\end{pmatrix}$

Does this converge absolutely? $= \sum_{\underbrace{(c,d)=1}} \frac{y^s}{|cz+d|^{2s}}$.

$T_\infty \backslash \Gamma = \{(c,d)=1\}$.

$$\sum_{c,d\geq 1} \frac{1}{(c^2+d^2)^s}$$ abs. converges iff $\boxed{\mathrm{Re}\, s > 1}$

Exercise: Prove this by polar coord integral in $\mathbb{R}^2$ using $\underline{r}\,dr\,d\theta$.

What does this have to do with Dirichlet?

Can we free condition $(c,d)=1$?

Mobius inversion.

$$\zeta(s) = \sum \frac{1}{n^s} = \prod_p \left(1-\frac{1}{p^s}\right)^{-1},$$

$$\frac{1}{\zeta}(s) = \prod_p \left(1-\frac{1}{p^s}\right) = \sum \frac{\mu(n)}{n^s}$$

Mobius

$$\mu(n) = \begin{cases} 0 & p^2 \mid n \\ (-1)^k & n = p_1 \cdots p_k \end{cases}$$

$$\sum_{n\geq 1} \frac{\boxed{1_{n=1}}}{n^s} = 1 = \zeta(s) \cdot \frac{1}{\zeta(s)} = \left(\sum_k \frac{1}{k^s}\right)\left(\sum_l \frac{\mu(l)}{l^s}\right)$$

$$\boxed{\frac{1}{2\pi i}\int_{(2)} F(s) X^s \, ds = \sum_{n<X} a_n}$$

$$\boxed{\sum_{d\mid n} \mu(d) = \begin{cases} 1 & n=1 \\ 0 & \text{else.} \end{cases}}$$

$$= \sum_n \frac{1}{n^s}\left(\sum_{kl=n} 1 \cdot \mu(l)\right).$$

More generally, $F(s) = \sum \frac{a_n}{n^s}$, $G(s) = \sum \frac{b_n}{n^s}$, Can define "Dirichlet convolution": $\boxed{\widehat{f*g} = \hat{f}\cdot\hat{g}}$,

$$\underbrace{F(s)}\cdot\underbrace{G(s)} = \sum_n \frac{(a*b)(n)}{n^s}, \quad (a*b)(n) = \sum_{d|n} a(d)\, b\left(\frac{n}{d}\right).$$

Back to $E(z,s) = \underbrace{\sum_{(c,d)\neq(0,0)}}\left(\frac{y^s}{|cz+d|^{2s}} \underbrace{\sum_{\substack{\ell|c\\ \ell|d}} \mu(\ell)}_{\ell|(c,d)}\right)$.

$(m,n)=\ell$. $\quad m/\ell = c,\ n/\ell = d$

Let $$\underbrace{E^*(z,s) = \sum_{(m,n)\neq(0,0)} \frac{y^s}{|mz+n|^{2s}}} = \underbrace{E(z,s)}\cdot \underbrace{\sum_{\ell\geq 1}\frac{1}{\ell^{2s}}}_{\zeta(2s)}.$$

Back towards class number formula.

Say $Q = [A,B,C] = Ax^2+Bxy+Cy^2$ $\quad \boxed{D = B^2-4AC < 0.}$

root $\alpha_Q = \dfrac{-B+\sqrt{|D|}\,i}{2A} \in \mathbb{H}$.

$$E^*(\alpha_Q, s) = \sum_{(m,n)\neq(0,0)} \left(\frac{\sqrt{|D|}}{2A}\right)^s \left(\frac{1}{|m\alpha_Q+n|^2}\right)^s.$$

$$|m\alpha_Q+n|^2 = \left(m\left(\frac{-B+\sqrt{|D|}\,i}{2A}\right)+n\right)\left(m\left(\frac{-B-\sqrt{|D|}\,i}{2A}\right)+n\right).$$

$$= \underbrace{\left(\frac{-mB}{2A}+n\right)^2} + \underbrace{\left(\frac{m\sqrt{|D|}}{2A}\right)^2}$$

$$= \frac{1}{4A^2}\left[\cancel{m^2B^2} - 4AnmB + 4A^2n^2 + m^2\left(4AC-\cancel{B^2}\right)\right]$$

$$= \frac{1}{\cancel{4A^2}}\left[\cancel{4}A^{\cancel{2}}n^2 - \cancel{4A}nmB + \cancel{4A}Cm^2\right]$$

$$= \frac{1}{A}Q(n,-m).$$

definite/$\mathbb{R}$ diagonalize to $\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}$.

$$E^*(\alpha_Q,s) = \sum_{(m,n)\neq(0,0)} \left(\frac{\sqrt{|D|}}{2\cancel{A}}\right)^s\left(\frac{\cancel{A}}{Q(n,+m)}\right)^s$$

drop.

$$= \frac{|D|^{s/2}}{2^s}\cdot\sum_{(m,n)\neq(0,0)}\frac{1}{Q(n,+m)^s}.$$

If $Q \sim Q'$ (properly / $SL_2(\mathbb{Z})$).

Epstein zeta function.

$\alpha_Q = \gamma \cdot \alpha_{Q'}$ for γ ψ.

$$E(\alpha_Q, s) = E(\alpha_{Q'}, s)$$

What's to come:

$$\sum_{[Q],\ \mathrm{disc}\ D} E(\overset{h(D).}{\alpha_Q}, s) = \zeta_K(s) \to L(\chi, 1).$$

Res $s = 1$.

Claim: $E(z,s)$ has mer'c cont & pole at $s=1$ $\left(= \frac{1}{\mathrm{vol}(\Gamma\backslash H)}\right)$.