Last time: $\sum_{k=1}^{N} k = 1+2+\cdots+N = \underbrace{\frac{N(N+1)}{2}}$.

$$S = 13 \overset{6}{\to} 19 \overset{6}{\to} 25 \overset{6}{\to} 31 + \cdots + 6001 = 999 \cdot 3007.$$

$$+S = 6001 + 5995 + \cdots + 13$$

$$2S = 6014 + \cdots + 6014 = 6014 \cdot \underbrace{999}.$$

13	$\overset{+6}{\to}$ 19	25	$\cdots$	6001
-7: 6	12	18	$\cdots$	5994
$\div 6$: 1	2	3	$\cdots$	999

$$S = \sum_{k=2}^{1000} (6k+1) = \sum_{k=1}^{999} (6k+7) = 7 \cdot 999 + 6 \cdot \sum_{k=1}^{999} k$$

$$= \sum_{k=1}^{999} 6k + \sum_{k=1}^{999} 7 \qquad 999 \cdot \frac{1000}{2}$$

$$= 1 \cdot 999 + 6\left[\left(\sum_{k=1}^{1000} k\right) - 1\right] = 999 \cdot 3007.$$

$$1000\left(\frac{1001}{2}\right) - 1$$

$$\sum_{k=500}^{1000} k = \left(\sum_{k=1}^{1000} k\right) - \left(\sum_{k=1}^{499} k\right) = \frac{1000 \cdot 1001}{2} - \frac{499 \cdot 500}{2},$$

$$1+4+9+16+\dots+625 = \sum_{k=1}^{25} k^2 = \frac{25\cdot 26\cdot 51}{6}.$$

$$\sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6}$$

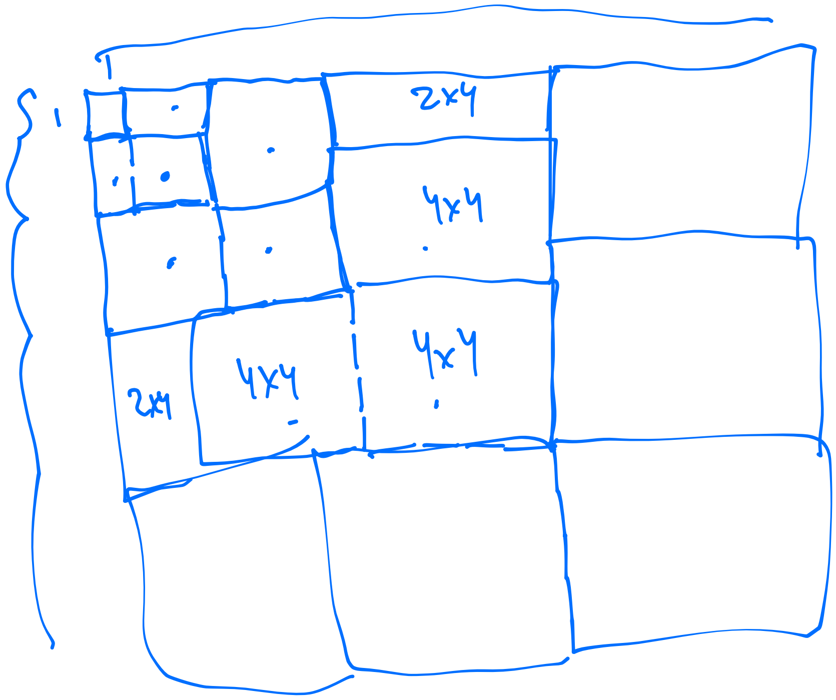
$$= 25\cdot 13\cdot 17.$$

$$\sum_{k=1}^{100} k^3 = 1+8+27+64+125+\dots+1000000 = \frac{100^2(101)^2}{4}.$$

$$\sum_{k=1}^{N} k^3 = \left(\sum_{k=1}^{N} k\right)^2 = \frac{N^2(N+1)^2}{4}$$

$$1\cdot 1^2+2\cdot 2^2+3\cdot 3^2+4\cdot 4^2+5\cdot 5^2 = (1+2+\dots+5)^2.$$

$1+2+3+4+5$



Recap:

$$\sum_{k=1}^{N} k = \frac{N(N+1)}{2}, \qquad \sum_{k=1}^{N} k^2 = \frac{N(N+1)(2N+1)}{6},$$

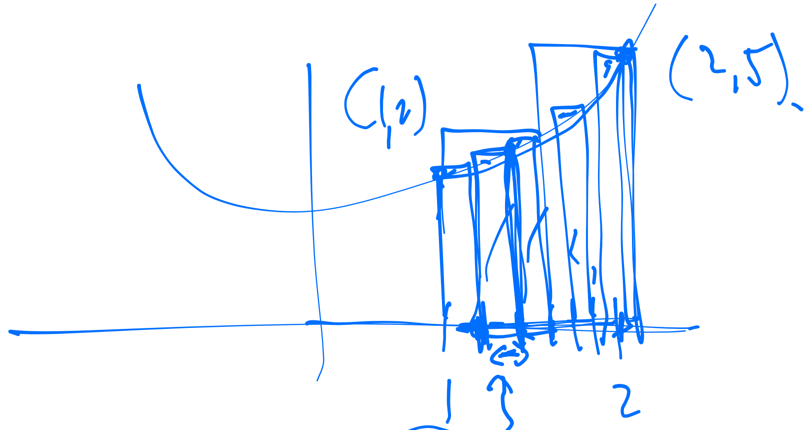
$$\sum_{k=1}^{N} k^3 = \frac{N^2(N+1)^2}{4}.$$

Last time:

Area "under"

$f(x) = 1 + x^2$

from $x = 1$ to 2



Divide into N pieces.

k'th location $x_k = 1 + \frac{1}{N} \cdot k$

$$\text{Area} \le \sum_{k=1}^{N} \frac{1}{N} \cdot f\left(1 + \frac{k}{N}\right)$$

$$= \sum_{k=1}^{N} \frac{1}{N} \cdot \left(1 + \left(1 + \frac{k}{N}\right)^2\right).$$

$$= \sum_{k=1}^{N} \left(\frac{2}{N} + \frac{2k}{N^2} + \frac{k^2}{N^3}\right)$$

$$= \frac{2}{N} \cdot N + \frac{2}{N^2}\left(\sum_{k=1}^{N} k\right) + \frac{1}{N^3} \sum_{k=1}^{N} k^2.$$

$$\text{Area} \le \underbrace{2 + \frac{2}{N^2} \cdot \frac{N(N+1)}{2} + \frac{1}{N^3} \frac{N(N+1)(2N+1)}{6}} = S(N)$$

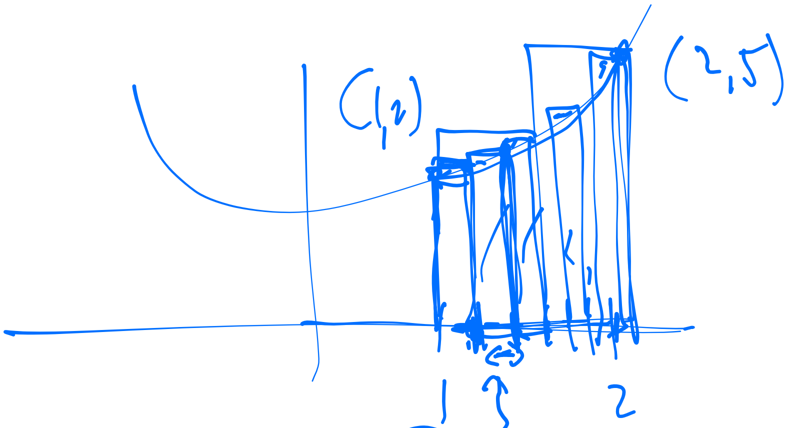
true for any N,

$$\xrightarrow[N \to \infty]{\text{as}} 2 + 1 + \frac{1}{3} = 3\tfrac{1}{3}, = \lim_{N \to \infty} S(N)$$

Area "under"

$f(x) = 1+x^2$

from $x=1$ to 2

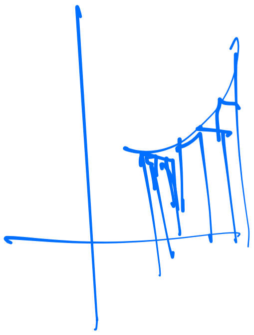


Divide into N pieces,

k'th position $x_k = 1 + \frac{1}{N}\cdot k$

$$\text{Area} \geq \sum_{k=0}^{N-1} \frac{1}{N}\cdot f\left(1+\frac{k}{N}\right)$$

$$= \sum_{k=0}^{N-1} \left(\frac{2}{N} + \frac{2k}{N^2} + \frac{k^2}{N^3}\right)$$



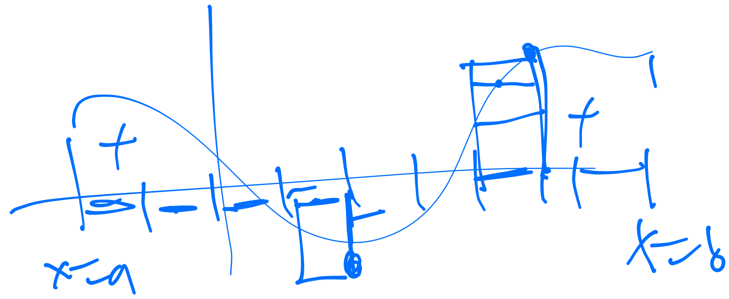
$$= \frac{2}{N}\cdot N + \frac{2}{N^2} \sum_{k=\cancel{0}\,1}^{N-1} k + \frac{1}{N^3}\cdot \sum_{k=\cancel{0}\,1}^{N-1} k^2$$

$$S(N) = 2 + \frac{2}{N^2}\,\frac{(N-1)\,N}{2} + \frac{1}{N^3}\cdot\frac{(N-1)(N-1+1)(2(N-1)+1)}{6}$$

$$\lim_{N\to\infty} S(N) = 2+1+\frac{1}{3} = 3\tfrac{1}{3}.$$

Given any nice function $f(x)$ on $a \leq x \leq b$

Want "area" "under" curve $y=f(x)$ from a to b.



$f\left(a+\frac{b-a}{N}\cdot 4\right)$

Break $[a,b]$ into N equal pieces of width $\frac{b-a}{N}$.

The height of kth rectangle: $f\left(a+\frac{b-a}{N}\cdot k\right)$.

Riemann sum: $S_f(N) = \sum_{k=1}^{N} \underbrace{\left(\frac{b-a}{N}\right)}_{\text{"dx"}} \cdot \underbrace{f\left(a+k\cdot\frac{(b-a)}{N}\right)}$.

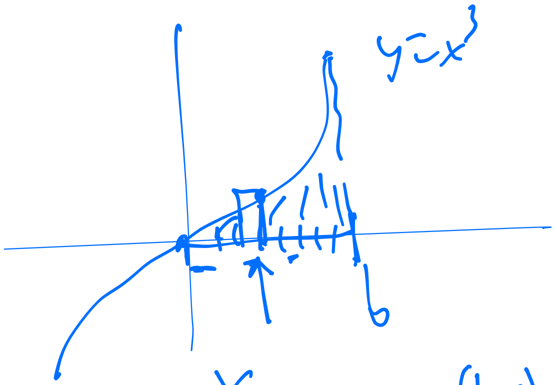
Def: Integral of f from a to b is:

$$\int_a^b \underbrace{f(x)}_{\substack{\uparrow \\ \text{height}}}\, \underbrace{dx}_{\substack{\uparrow \\ \text{width}}} = \lim_{N\to\infty} S_f(N).$$

$$\boxed{\begin{array}{c}\text{Yoga} \\ \sum \xrightarrow[N\to\infty]{} \int\end{array}}$$

Area from above: $\int_1^2 (1+x^2)\cdot dx$

Ex: $\int_{a=0}^{b} x^3\, dx = \lim_{N\to\infty} S(N)$



$$S(N) = \sum_{k=1}^{N} \frac{b}{N}\left(\frac{k\cdot b}{N}\right)^3$$

$$x_k = a + k\cdot\frac{(b-a)}{N} = \frac{k\cdot b}{N}.$$

$$= \frac{b^4}{N^4}\sum_{k=1}^{N} k^3$$

$$\boxed{\cancel{\frac{b^4}{\infty^4}\sum_{k=1}^{\infty} k^3}}$$

$$= \frac{b^4}{N^4}\left(\frac{N^2\,(N+1)^2}{4}\right) \xrightarrow[\text{as } N\to\infty]{} \frac{b^4}{4} = \int_0^b x^3\,dx.$$

Ex: Recognize this limit without computing it:

$$\lim_{N\to\infty} \frac{3-(-7)}{N}\sum_{k=1}^{N} e^{\overset{-7}{\cancel{3}}+\frac{k10}{N}}\cdot \cancel{e^{-10}}\,e^{10}$$

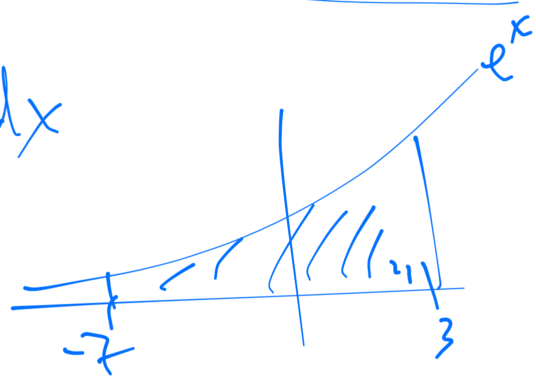
$$\boxed{\text{Riemann Sum: } S(N) = \sum_{k=1}^{N}\frac{b-a}{N}\cdot f\left(a+\frac{k(b-a)}{N}\right)}.$$

$$= e^{10}\int_{-7}^{3} e^x\,dx$$

$b=3$

$a=-7$

$f(x)=e^x$



Ex: Recognize:

$$\lim_{N\to\infty} \sum_{k=1}^{N} \frac{6-3}{N}\left(\tan\left(\frac{2\pi}{3+\frac{3k}{N}}\right) + \left(3+\frac{3k}{N}\right)^5\right).$$

$$= \int_3^6 \left(\tan\left(\frac{2\pi}{x}\right) + x^5\right)\cdot dx$$

1 $b = 3.2$ −10 10

2 $a = -1.5$ −10 10

3 $f(x) = 1 + x^2 + \sin(2\pi x)$

4 $y = f(x)\ \{a \le x \le b\}$

5 $N = 18$ 1 100

6 $k = [1 \ldots N]$

k = 18 element list

7 $0 \le y \le f\left(a + \frac{k(b-a)}{N}\right)\left\{a + \frac{(k-1)(b-a)}{N} \le x \le a + \frac{k(b-a)}{N}\right\}$

8 $\int_a^b f(x)dx$

= 16.5393301414

9 $\sum_{j=1}^{N} \frac{b-a}{N} f\left(a + \frac{j}{N}(b-a)\right)$

= 17.8090118535

10