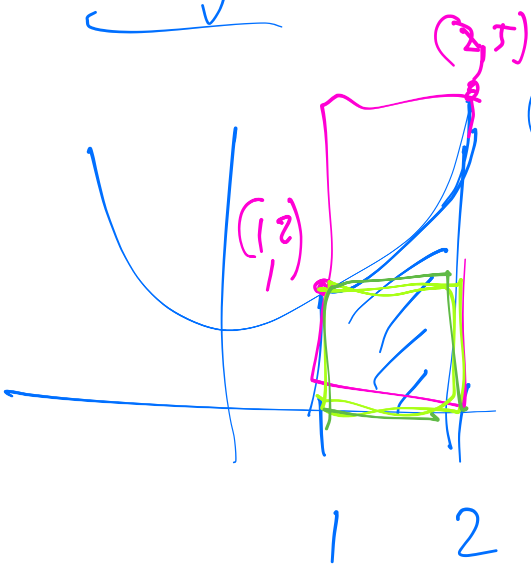
Recall: tangent Slope = "derivative" $f'(x) = \lim_{u \to x} \frac{f(u)-f(x)}{u-x}$

$\frac{d}{dx}(x^n) = nx^{n-1}$, $\frac{d}{dx}$ sin/cos/ e^x/ $\ln x$, ;

Product/Quotient/Chain "Rule" = shortcuts

§ Area = "integral"

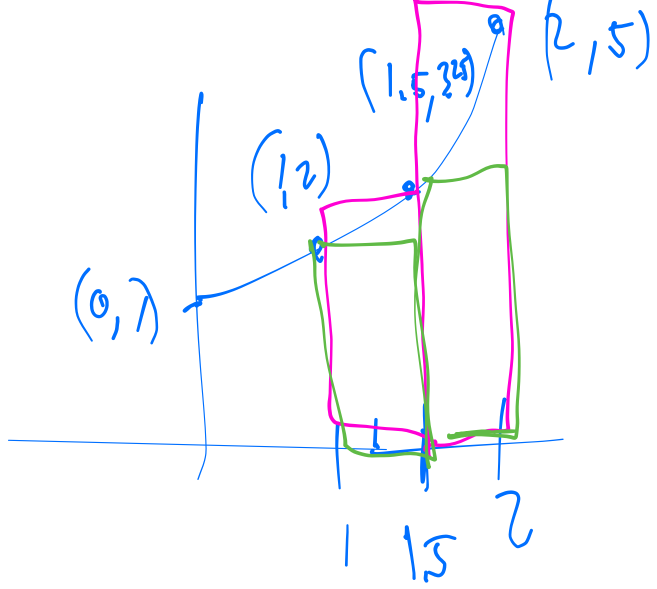
Eg: $f(x) = 1+x^2$, from $x=1$ to $x=2$.



Goal: Area = ?.

$\underbrace{1 \cdot 5}_{5} \geq \text{Area} \geq 1 \cdot 2 = 2$,

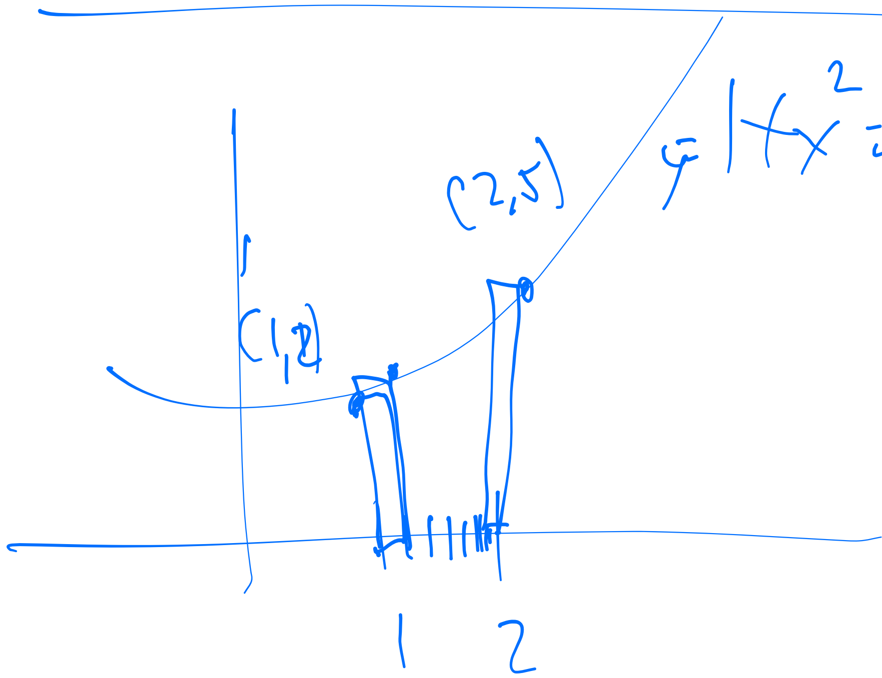
Better:



$$f\left(\tfrac{3}{2}\right) = 1 + \tfrac{9}{4} = 3.25$$

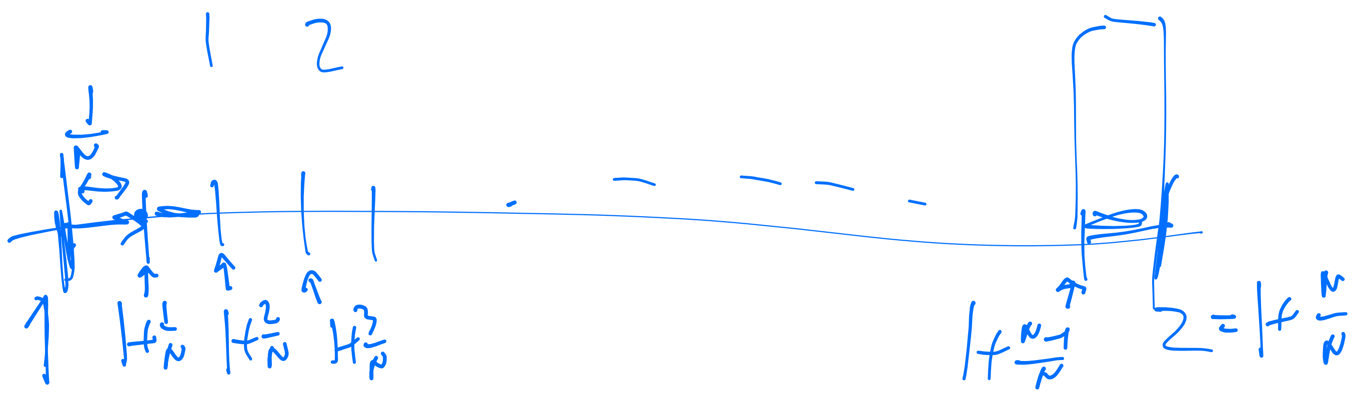
$$\underbrace{\tfrac{1}{2}\cdot 5 + \tfrac{1}{2}\cdot 3.25}_{4.125} \geq \text{Area} \geq \underbrace{\tfrac{1}{2}\cdot 2 + \tfrac{1}{2}\cdot 3.25}_{2.625}$$

$\boxed{\text{Idea: Break hard problem into } \infty\text{ly many } \infty\text{ly easy problems, add all } \infty \text{ of them up.}}$



$y = 1 + x^2 = f(x)$

Break into N pieces.



Upper sum:

$$\underbrace{\frac{1}{N}\cdot\left[f\left(1+\frac{1}{N}\right)\right]+\overbrace{\frac{1}{N}\left[f\left(1+\frac{2}{N}\right)\right]}^{k=2}+\overbrace{\frac{1}{N}f\left(1+\frac{3}{N}\right)}^{k=3}+\dots+\overbrace{\frac{1}{N}\cdot f\left(1+\frac{N}{N}\right)}^{k=N}}_{\sum_{k=1}^{N}\frac{1}{N}\cdot f\left(1+\frac{k}{N}\right)} > \text{Area}$$

"Sigma = sum"

Quiz: $\sum_{k=3}^{7} k^2 \sin(k) = ?$ (without Σ notation)

$$= \underline{3^2\sin(3)} + \underline{4^2\sin(4)} + \underline{5^2\sin 5} + \underline{6^2\sin 6} + \underline{7^2\sin 7}$$

$f(x) = 1+x^2$

$$\left[1+\left(1+\frac{1}{N}\right)^2\right] \qquad \left[1+\left(1+\frac{3}{N}\right)^2\right] \qquad \left[1+\left(1+\frac{N}{N}\right)^2\right]$$

$$\frac{1}{N}\cdot\left[f\left(1+\frac{1}{N}\right)\right]+\frac{1}{N}\left[f\left(1+\frac{2}{N}\right)\right]+\frac{1}{N}f\left(1+\frac{3}{N}\right)+\dots+\frac{1}{N}f\left(1+\frac{N}{N}\right) > \text{Area}$$

$$=\sum_{k=1}^{N}\frac{1}{N}\cdot f\left(1+\frac{k}{N}\right)=\sum_{k=1}^{N}\frac{1}{N}\left[1+\left(1+\frac{k}{N}\right)^2\right]$$

$$f(x)=1+x^2 \qquad [1,2]$$

$$=\sum_{k=1}^{N}\frac{1}{N}\left[1+1^2+2\frac{k}{N}+\frac{k^2}{N^2}\right].$$

$$=\sum_{k=1}^{N}\frac{2}{N}+\frac{2k}{N^2}+\frac{k^2}{N^3}=\left(\frac{2}{N}+\frac{2\cdot 1}{N^2}+\frac{1^2}{N^3}\right)+\left(\frac{2}{N}+\frac{2\cdot 2}{N^2}+\frac{2^2}{N^3}\right)+\dots(\quad)$$

$$=\sum_{k=1}^{N}\frac{2}{N}+\sum_{k=1}^{N}\frac{2k}{N^2}+\sum_{k=1}^{N}\frac{k^2}{N^3}.$$

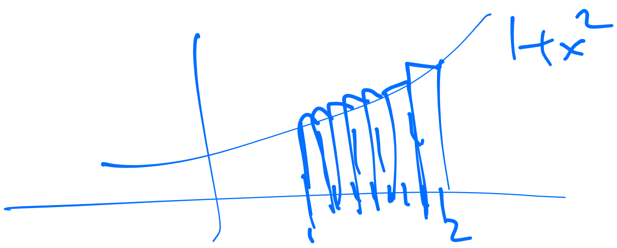
"Rule":
$$\sum_{k=1}^{N}\left[g(k)+h(k)\right]=\sum_{k=1}^{N}g(k)+\sum_{l=1}^{N}h(l).$$

$$\hookrightarrow = \underbrace{\left(\frac{2}{N}+\frac{2}{N}+\cdots\frac{2}{N}\right)}_{N \text{ times}} + \frac{2}{N^2}(1+2+3+\cdots+N) + \frac{1}{N^3}\sum_{k=1}^{N} k^2 .$$

"Rule": $$\sum_{u=a}^{b} c = c(b-a+1)$$

"Rule": $$\sum_{w=1}^{N} c\cdot g(w) = c\cdot\sum_{w=1}^{N} g(w).$$

Recap so far:



For any N,

$$N\cdot\frac{2}{N} + \frac{2}{N^2}\sum_{k=1}^{N} k + \frac{1}{N^3}\sum_{k=1}^{N} k^2 \geq \text{Area}$$

$$\sum_{k=1}^{N} \frac{2k}{N^2} = \frac{2\cdot 1}{N^2} + \frac{2\cdot 2}{N^2} + \frac{2\cdot 3}{N^2} + \frac{2\cdot 4}{N^2} + \cdots + \frac{2\cdot N}{N^2}$$

$$= \frac{2}{N^2}(1+2+3+\cdots+N) = \frac{2}{N^2}\cdot\sum_{k=1}^{N} k$$

Gauss:

$$\begin{aligned} S &= \overbrace{1+2+3+4+\dots+1000}\\ + \quad S &= 1000+999+998+\dots+2+1 \end{aligned}$$

$$2S = 1001+1001+1001+\dots+1001$$

$$= 1{,}001{,}000$$

$$S = 500{,}500.$$

$$S = 1+2+3+\dots+N$$

$$+ \quad S = N+(N-1)+(N-2)+\dots 2+1$$

$$2S = (N+1)+(N+1)+(N+1)\dots+(N+1) = N(N+1)$$

$$S = \frac{N(N+1)}{2}.$$

Δ - numbers.

$1=1$, $1+2=3$, $1+2+3=6$, $1+2+3+4=10$

$\frac{2}{2}(2+1)$ // $\frac{3(4)}{2}$ // $\frac{4(5)}{2}$ //

Thm: $1+2+3+\dots+N = \sum_{k=1}^{N} k = \frac{N(N+1)}{2}$.

Thm: $\boxed{\sum_{k=1}^{N} k^2} = 1+4+9+16+\dots+N^2$

$$\boxed{= \frac{N(N+1)(2N+1)}{6}}$$

$1=1$, $1+4=5$, $1+4+9=14$, $1+4+9+16=30$, ...

$$5 = \frac{2(3)5}{6}$$

$1+8 = 9 = (1+2)^2$ $\qquad$ $1+8+27+64 = 100 = (1+2+3+4)^2$

$1+8+27 = 36 = (1+2+3)^2$

$$\sum_{k=1}^{N} k^3 = \left(\sum_{k=1}^{N} k \right)^2$$