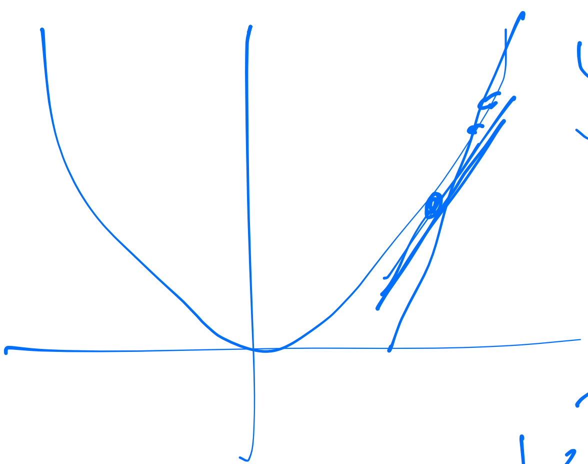
Calculus off the beaten path.

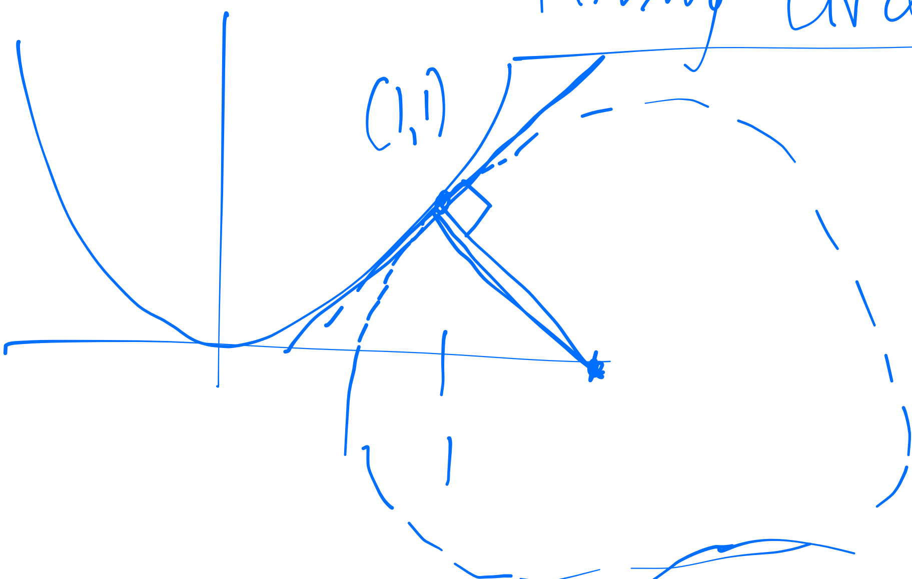


$y = x^2$

Fermat 1630s?

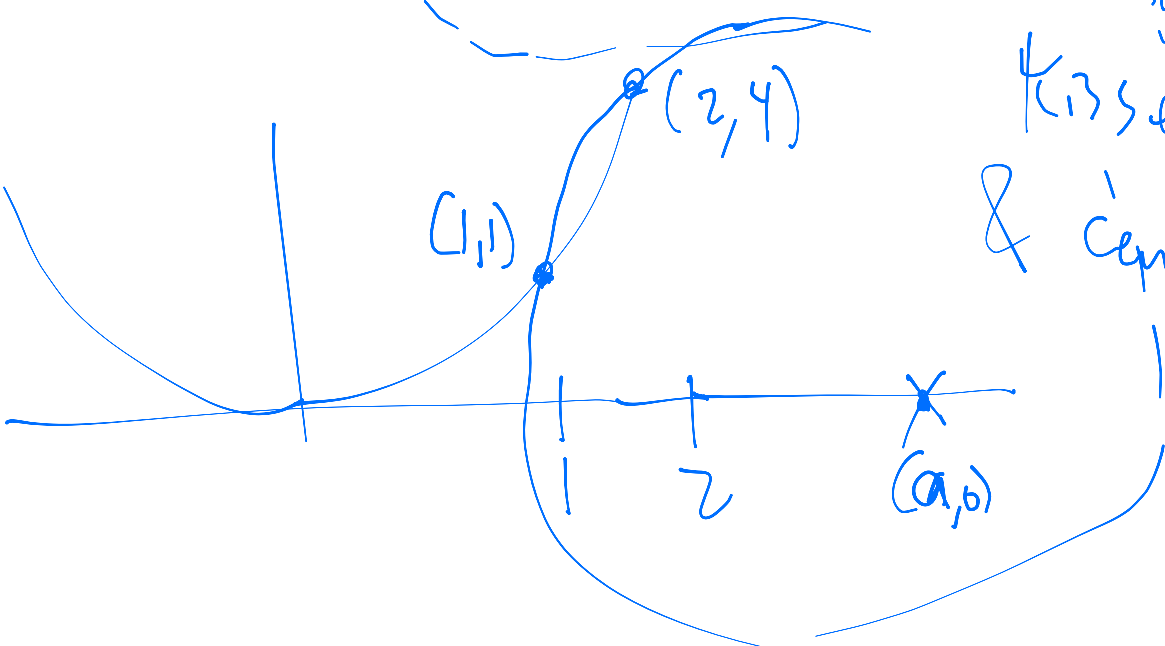
Descartes

Kissing Circles



Secant lines → tangent line.

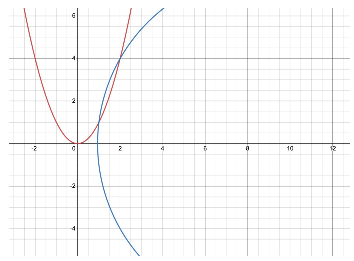
Idea: Find magical circle kisses the graph & centered on x-axis



Find a & r so that $(x-a)^2+(y-\cancel{b})^2=r^2$ contains (1,1), (2,4)

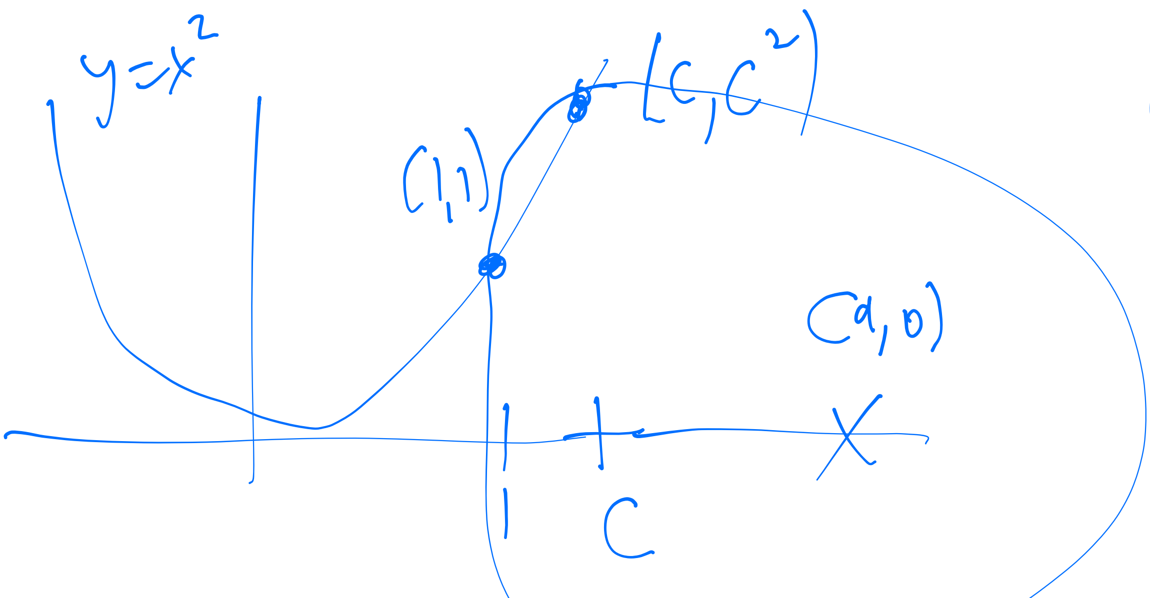
$$\begin{cases} (1-a)^2 + 1^2 = r^2 \\ (2-a)^2 + 4^2 = r^2 \end{cases} \Rightarrow 1 - 2a + \cancel{a^2} + 1 = 4 - 4a + \cancel{a^2} + 16,$$

$$\Rightarrow \quad 2a = 18, \quad a = 9$$



$$64 + 1^2 = r^2, \quad r = \sqrt{65}.$$

Again, want circle thru

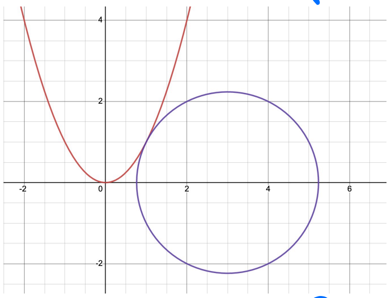


$$(x-a)^2 + y^2 = r^2$$

$$\begin{cases} (1-a)^2 + 1 = r^2 \\ (c-a)^2 + c^4 = r^2 \end{cases}$$

$$1 - 2a + \cancel{a^2} + 1 = c^2 - 2ca + \cancel{a^2} + c^4$$

$$a(2c-2) = c^4 + c^2 - 2.$$

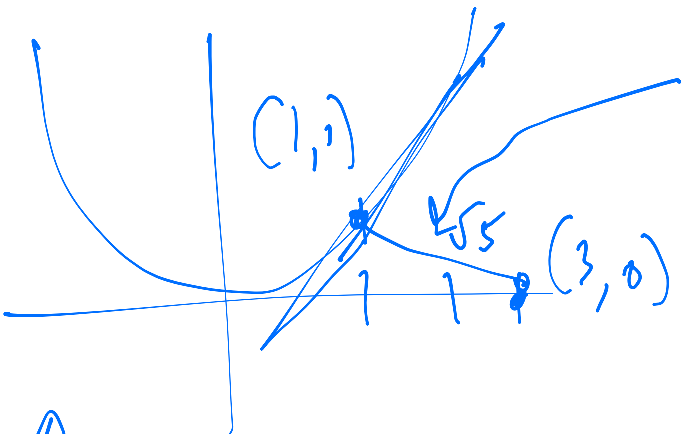


$$a = \frac{c^4+c^2-2}{(2c-2)}.$$

$$r^2 = (1-a)^2 + 1^2$$

Finally, take limit as $c \to 1$

$$\lim_{c\to1} a = \lim_{c\to1} \frac{\overset{\to 0}{c^4+c^2-2}}{\underset{\to 0}{2c-2}} \overset{(l'H)}{=} \lim_{c\to1} \frac{4c^3+2c}{2} = 3.$$

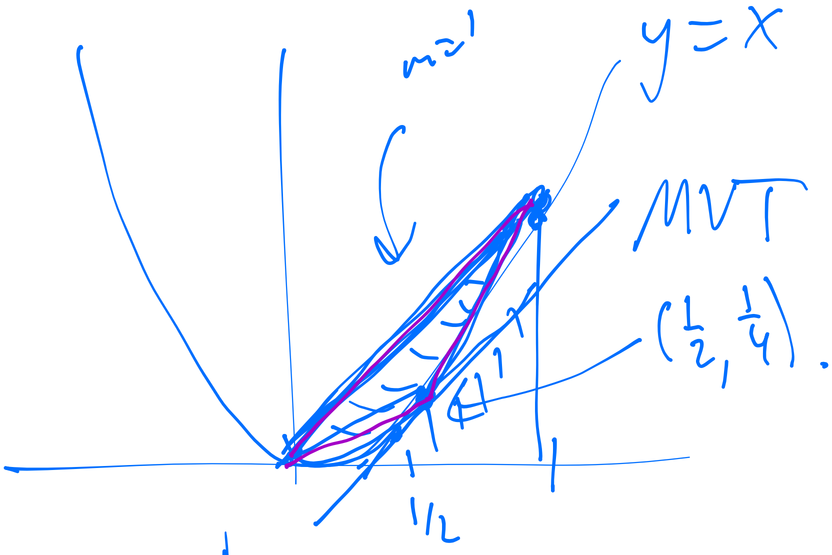


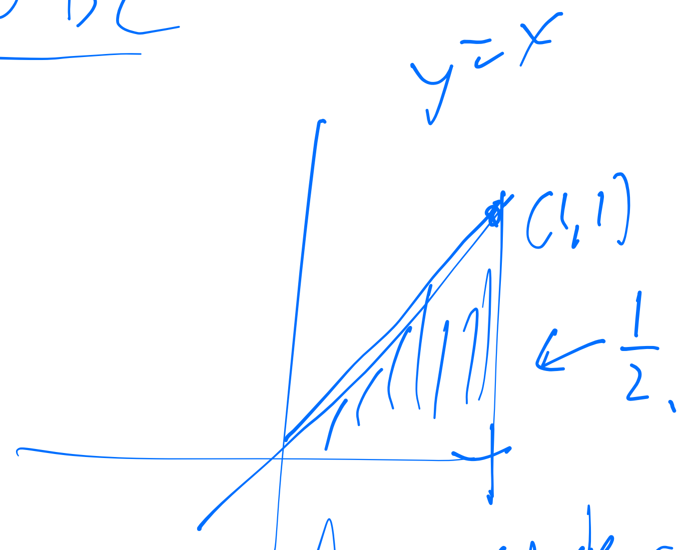
$m = \frac{-1}{2}$, $m_1 = 2$;)

$$\frac{d}{dx}x^2\Big|_{x=1} = 2x\Big|_{x=1} = 2$$

Area ("Integral").

Archimedes ~ 250 BC

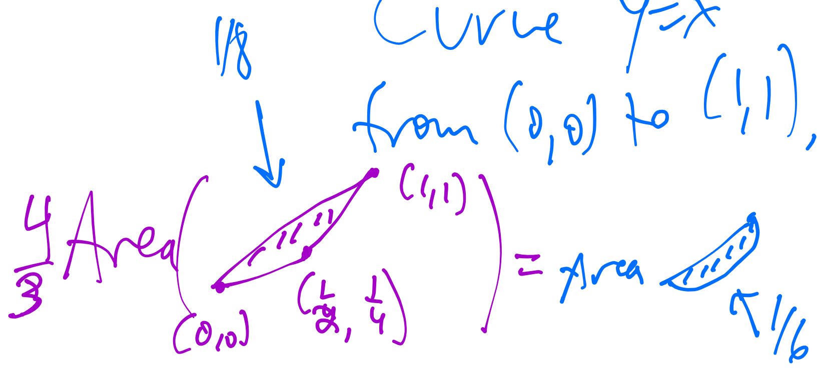




Area under Curve $y=x$ from $(0,0)$ to $(1,1)$,

$$\frac{d}{dx}x^2 = 2x = 1$$

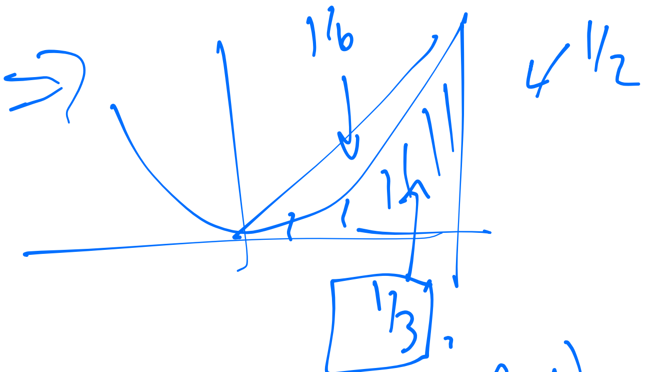
$$x = \frac{1}{2}$$



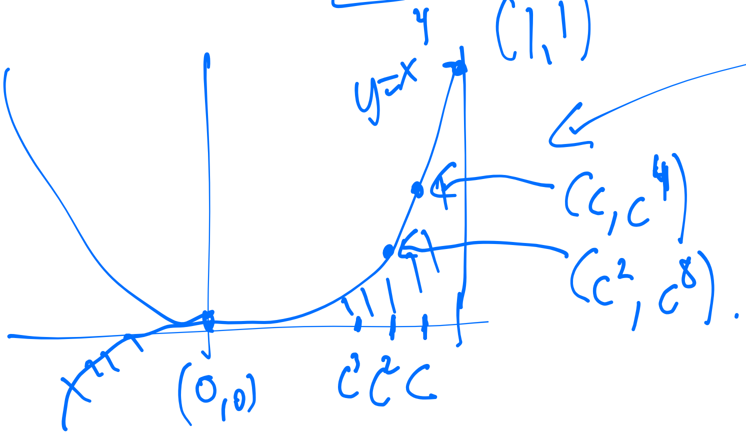
Quadrature of Parabola.

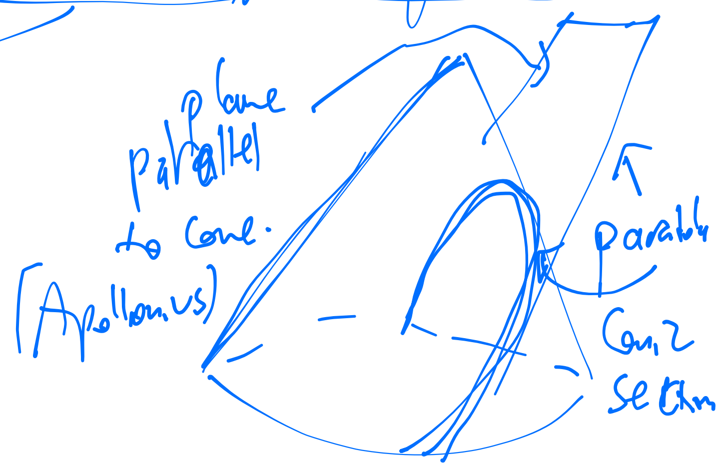
"Saving" = figure out area.

⇒

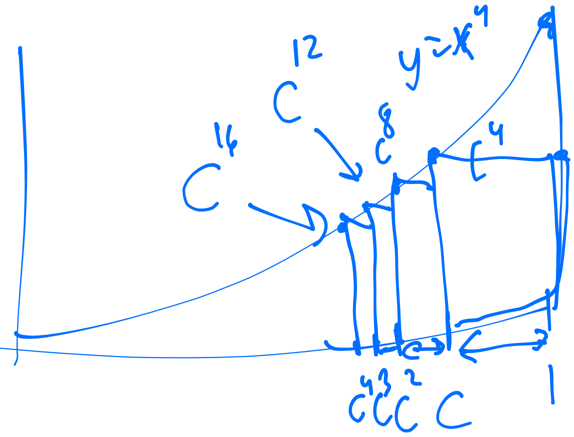


Fermat's Method for area under $y=x^n$.





Fix $c<1$, c^2



$$\text{Area} \geq (1-c)c^4 + \underbrace{\cancel{(c-c^2)}}_{c(1-c)}c^8 + \underbrace{\cancel{(c^2-c^3)}}_{c^2(1-c)}c^{12} + \underbrace{\cancel{(c^3-c^4)}}_{c^3(1-c)}c^{16} + \dots$$

$$= (1-c)c^4\left[1 + c^5 + c^{10} + c^{15} + c^{20} + \dots\right].$$

$$(1+r+r^2+r^3+r^4+r^5+\dots)(\underline{1-r}) = 1$$

$$= 1 + r + r^2 + r^3 + \dots \quad (|r|<1).$$
$$\quad - r - r^2 - r^3 - \dots$$

$$\boxed{1+r+r^2+r^3+\dots = \frac{1}{1-r}}$$

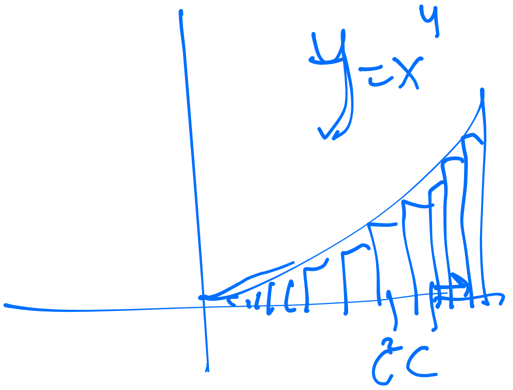
Complex analysis

$$1+2+4+8+16+\dots \text{ "=" } \frac{1}{1-2} = -1.$$

Fix any $c<1$ $\qquad r = c^5 < 1.$

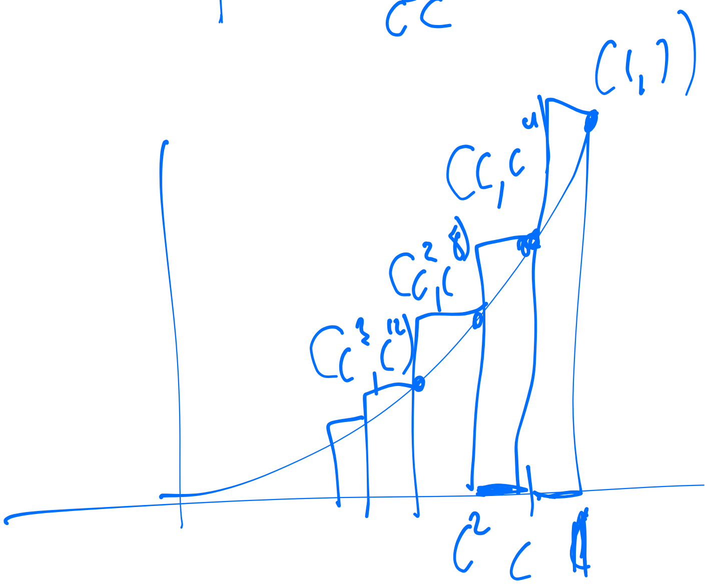
$$\text{Area} \geq (1-c)c^4\big[\underbrace{1 + c^5 + c^{10} + c^{15} + c^{20} + \cdots}\big].$$

$$= (1-c)c^4 \underbrace{\frac{1}{1-c^5}} = \frac{c^4-c^5}{1-c^5}$$



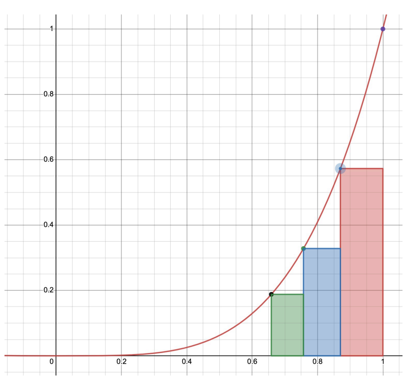
$$\text{Area} \geq \lim_{c\to 1^-} \frac{c^4-c^5}{1-c^5} \underset{(l'H)}{=} \lim_{c\to 1^-} \frac{4c^3-5c^4}{-5c^4}$$

$$= \boxed{\frac{1}{5}}$$



$$\text{Area} \leq (1-c)\cdot 1 + (c-c^2)\,c^4 + (c^2-c^3)\cdot c^8 + \cdots$$

$$= (1-c)\big[1 + c^5 + c^{10} + c^{15} + c^{20} + \cdots\big]$$



$$= \frac{1-c}{1-c^5} = \frac{1}{1+c+c^2+c^3+c^4} \xrightarrow{\text{as } c\to 1^-} \frac{1}{5}$$

$$(1-c^5) = (1-c)(1+c+c^2+c^3+c^4)$$

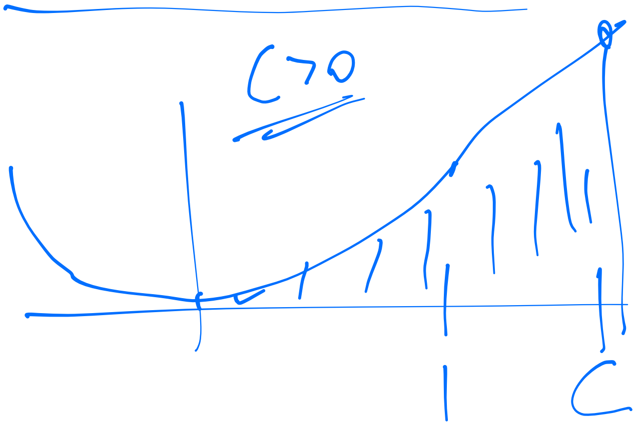
Fermat: Area under $(y = x^4)$ from $x=0$ to $x=1$

is $\frac{1}{5}$.

Archimedes: Area under $(y=x^2)$ from 0 to 1 is $\frac{1}{3}$

In general: area under $y=x^n$ from $x=0$ to $x=1$ is $\frac{1}{n+1}$

Isaac Barrow (Newton's teacher).



Area under $y=x^n$ from $x=0$ to $x=C$

is $\boxed{\frac{1}{n+1} \cdot C^{n+1}}$.

Antideriv of x^n is $\boxed{\frac{x^{n+1}}{n+1}} + K$!!!

UK Newton (22) 1665 home from Cambridge 1690→1710s,

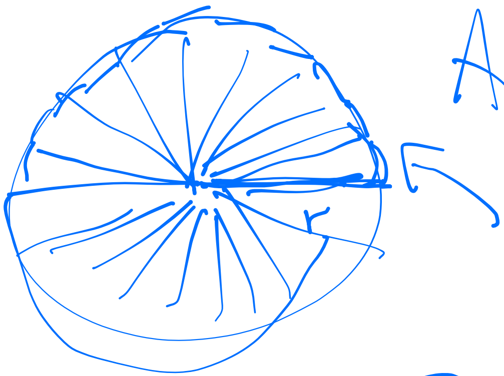
$\dot{x}$ $\ddot{x}$

General functions: integral calculus. (Fluxions),

Principia

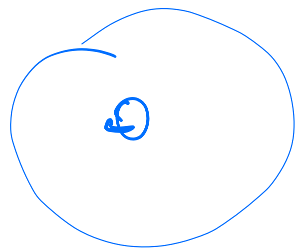
$\frac{d^5}{dx^5}$ Leibnitz Europe 1674 (re)discovers calculus publishes 1680s, J. Bernoulli (Basel), Euler

Madhava (1400s).

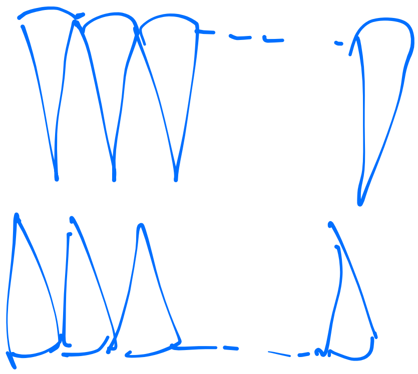


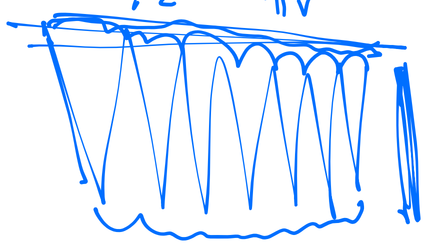
Archimedes, $A = \pi r^2$.

$\pi = \frac{\text{circumf}}{\text{diam.}}$



$C = 2\pi r$





$C/2 = \pi r$

r

Newton's new computation of π (also due to Madhava).

Pascal's triangle

0 1 1 0

0 1 2 1 0

0 1 3 3 1 0

0 1 4 6 4 1 0

1 5 10 10 5 1 0

$11^3 = 1331$

$11^4 = 14641$.

$(x=10)$.

$(1+x)^2 = 1+2x+x^2$

$(1+x)^3 = 1+3x+3x^2+x^3$

$(1+x)^5 = 1+5x+10x^2+10x^3$

$+5x^4+1\cdot x^5$

$$1+5x+10x^2+10x^3+5x^4+x^5$$

$$=1+\frac{5}{1}x+\frac{5\cdot4}{2\cdot1}x^2+\frac{5\cdot4\cdot3}{3\cdot2\cdot1}x^3+\frac{5\cdot4\cdot3\cdot2}{4\cdot3\cdot2\cdot1}x^4$$

$$+\frac{5\cdot4\cdot3\cdot2\cdot1}{5\cdot4\cdot3\cdot2\cdot1}x^5$$

$$\cancel{+\frac{5\cdot4\cdot3\cdot2\cdot1\cdot0}{6!}x^6}$$

Newton: $$(1+x)^{-1}=1+\frac{(-1)}{1}x+\frac{(-1)(-2)}{2!}x^2+\frac{(-1)(-2)(-3)}{3!}x^3+\frac{(-1)(-2)(-3)(-4)}{4!}x^4+\ldots$$

$$=1-x+x^2-x^3+x^4-x^5+\ldots \checkmark$$

$$1+r+r^2+r^3+\ldots=\frac{1}{1-r},\quad r=-x.$$

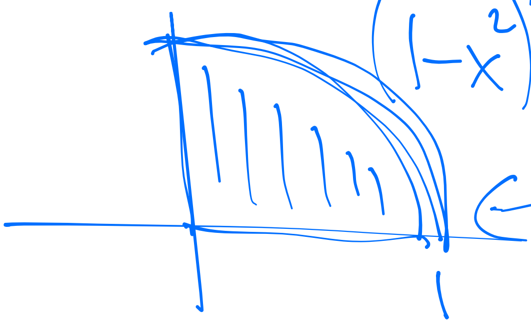
$$(1+x)^{1/2}=1+\frac{1/2}{1}\cdot x+\frac{1/2\left(-\frac{1}{2}\right)}{2!}x^2+\frac{1/2\left(-1/2\right)\left(-3/2\right)}{3!}x^3+\ldots$$

$$\left(1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3+\cdots\right)\left(1+\frac{1}{2}x-\frac{1}{8}x^2+\frac{1}{16}x^3+\cdots\right)$$

$$= 1+2\cdot\frac{1}{2}x+\left(-\frac{1}{8}\right)x^2+\left(-\frac{1}{8}x^2\right)+\frac{1}{4}x^2$$

$$+\frac{2}{16}x^3+2\left(\frac{1}{2}x\right)\left(-\frac{1}{8}x^2\right)+0+0.$$

$$(1-x^2)^{1/2} = 1-\frac{x^2}{2}+\frac{x^4}{8}-\cdots$$



Area $= \boxed{\frac{\pi}{4}}$