Review Sheet for Math 151 Midterm 3 Fall 2022

The following questions are intended to give you practice working problems that test the ideas covered in the course for the third midterm exam. The number of problems on this review sheet is larger than the number of problems on a typical 80-minute exam – this sheet is NOT A PRACTICE TEST. You should not memorize the problems, as the problems you encounter on the midterm exam will not look exactly like the problems on this review sheet – the exam problems will be different. Your goal is to be able to think through and understand the processes required to answer the questions correctly. Before you work the review problems, you should study for the exam and when you feel you have prepared enough, try doing the problems on this sheet WITHOUT looking at your notes, textbook, or videos. Make sure to try this a couple of days before the midterm so that you will have time to fill in any gaps of knowledge you uncover. If you start your studying by doing the review sheet first, you will not maximize the benefit of the review sheet.

1. Use an appropriate linearization to approximate $\frac{1}{\sqrt{10}}$.

   (a) What would be an appropriate function $f(x)$ and center a for this linearization?

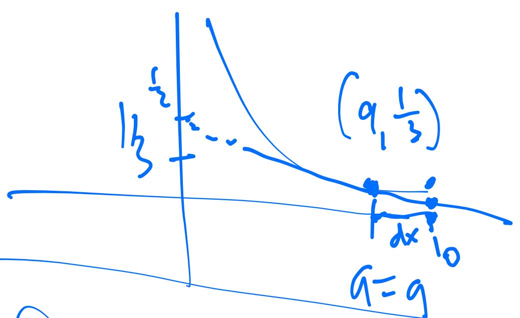
   (b) Find $L(x)$ for your choices from part (a).

   (c) Find dx so that $f(a + dx) = \frac{1}{\sqrt{10}}$.

   (d) Find the standard linear approximation of $\frac{1}{\sqrt{10}}$ using your choices for f and a.

(a) $\boxed{f(x) = \frac{1}{\sqrt{x}}, \quad a = 9,}$ $f(9) = \frac{1}{\sqrt{9}} = \frac{1}{3}$,

$f'(x)\Big|_{a=9} = -\frac{1}{2}x^{-3/2}\Big|_{a=9} = -\frac{1}{2}\frac{1}{9^{3/2}} = -\frac{1}{54}$



$L(x)$

$y - \frac{1}{3} = -\frac{1}{54}(x - 9)$

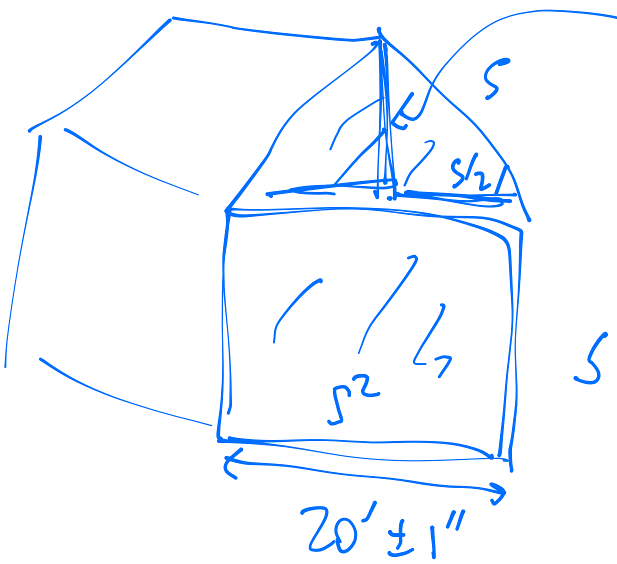
(b) $\boxed{L(x) = -\frac{1}{54}x + \frac{1}{2}.}$

(c) $\boxed{dx = 1}$, $f(9+1) = \frac{1}{\sqrt{10}}$

(d) $f(10) \approx L(10) = \boxed{-\frac{10}{54} + \frac{1}{2}}$

2. One side of a house has the shape of a square surmounted by an equilateral triangle (i.e., the equilateral triangle sits on top of the square). If the base is measured as 20 ft, with a maximum error in measurement of 1 in, use differentials to approximate the relative error in the calculation of the area of the side of the house if the measurement of 20 ft for the base is used.

percent



$$\text{Total area} = s^2 + \frac{\sqrt{3}}{4}s^2 = \frac{4+\sqrt{3}}{4}s^2$$

$$A = \frac{4+\sqrt{3}}{4}s^2,$$

$$dA = \frac{4+\sqrt{3}}{4} \cdot 2s \cdot ds \qquad s = 20', \quad ds = \tfrac{1}{12}'$$

$$\boxed{dA = \frac{4+\sqrt{3}}{4} \cdot 2 \cdot 20 \cdot \frac{1}{12}\ ft^2.}$$

$$\boxed{\frac{dA}{A}} = \text{percent error} = \frac{\cancel{\frac{4+\sqrt{3}}{4}} \cdot 2 \cdot \cancel{20} \cdot \frac{1}{12}}{\cancel{\frac{4+\sqrt{3}}{4}} \cdot \cancel{20} \cdot 20} = \boxed{\frac{1}{120}.}$$

3. Consider the function

$$f(x) = -12x^{1/3}(2x^2 + 14x - 70).$$

Use f in parts (a)-(c).

(a) Find the critical points of f.

(b) Find the local maxima and the locations where the local maxima occur.

(c) Find the local minima and the locations where the local minima occur.

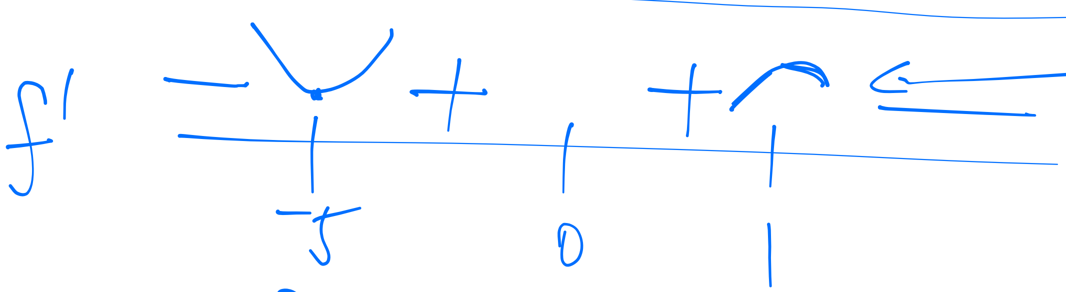
$$f'(x) = -12\left[x^{1/3}(4x+14) + \frac{1}{3}x^{-2/3}(2x^2+14x-70)\right]$$

$$= \frac{-12x^{-2/3}}{3}\left[3x(4x+14) + (2x^2+14x-70)\right].$$

$$= -4x^{-2/3}\left[14x^2 + 14\cdot 4x - 14\cdot 5\right]$$

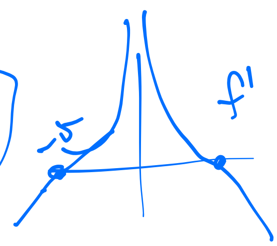
$$= -4\cdot x^{-2/3}\cdot 14\,(x+5)(x-1)$$

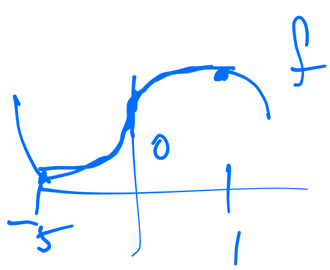
a) $\boxed{\text{Critical values: } x = 1, -5 \ \& \ x = 0.}$



(b) local max at $x=1$

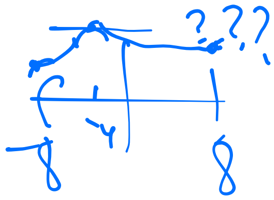
(c) local min at $x=-5$





4. Find the absolute maximum and minimum values and where they occur for the function

$$f(x) = \sin^{-1}(x/8) - \frac{1}{4}\sqrt{64 - x^2} \text{ on the interval } [-8, 8].$$



$$f'(x) = \frac{1}{\sqrt{1-\left(\frac{x}{8}\right)^2}} \cdot \frac{1}{8} + \frac{1}{4}\left(\frac{1}{2}\right)\left(64-x^2\right)^{-1/2}(2x)$$

$$\frac{1}{8\sqrt{1-\left(\frac{x}{8}\right)^2}} = \frac{1}{\sqrt{64-8^2\left(\frac{x}{8}\right)^2}}$$

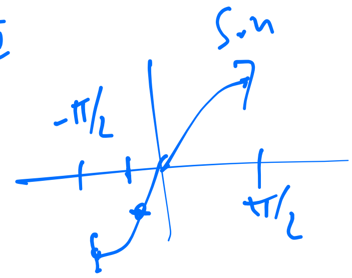
$$= \frac{1}{\sqrt{64-x^2}}\left[1 + \frac{x}{4}\right],$$

Crit values: $x = -4$

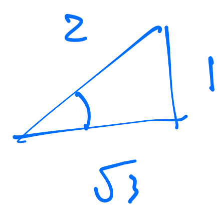
$x = 8, -8.$

$$f(8) = \sin^{-1}(1) - \frac{1}{4}\sqrt{0} = \frac{\pi}{2} \approx \frac{3}{2}$$

$$f(-8) = \sin^{-1}(-1) - 0 = -\frac{\pi}{2} \approx -\frac{3.1}{2}$$



$$f(-4) = \sin^{-1}\left(-\frac{1}{2}\right) - \frac{1}{4}\sqrt{64-16} =$$



$$= -\frac{\pi}{6} - \frac{1}{4}\cdot 4\sqrt{3}$$

$$= -\frac{\pi}{6} - \sqrt{3} \approx -\frac{1}{2} - 1.4 = -\frac{3}{2}$$

$$\boxed{-\frac{\pi}{6} - \sqrt{3}} < -\frac{\pi}{2}$$

$< \frac{\pi}{2}$

$$\boxed{\text{Abs max} = \frac{\pi}{2}, \text{ at } x = 8}$$

$$\boxed{\text{Abs min} = -\frac{\pi}{6} - \sqrt{3}, \text{ at } x = -4}$$

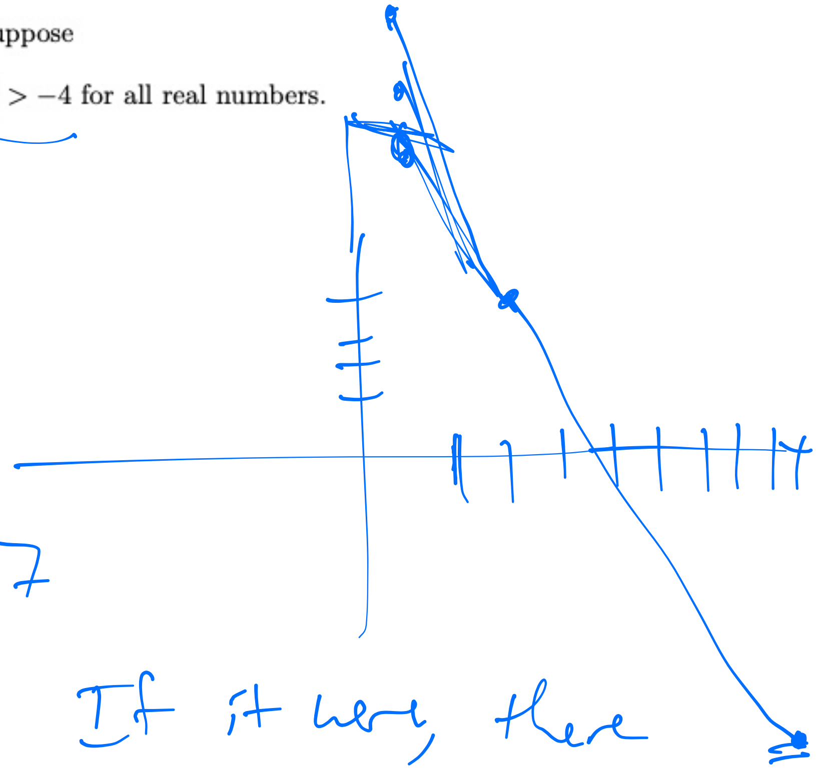
5. Let f be a differentiable function. Furthermore, suppose

$$f(2) = 4 \qquad \text{and} \qquad f'(x) > -4 \text{ for all real numbers.}$$

Use this information in parts (a) and (b).

(a) What can we say about the value of $f(7)$?

(b) What can we say about the value of $f(1)$?



Avg rate of change from $x=2$ to $x=7$ cannot be ≤ -4. If it were, there would be some $c \in (2,7)$ with $f'(c) \leq -4$ by MVT.

$$\frac{f(7)-f(2)}{7-2} > -4$$

$$\Rightarrow \quad f(7) - \overset{4}{\cancel{f(2)}} > -20 \Rightarrow \boxed{f(7) > -16}$$

(a)

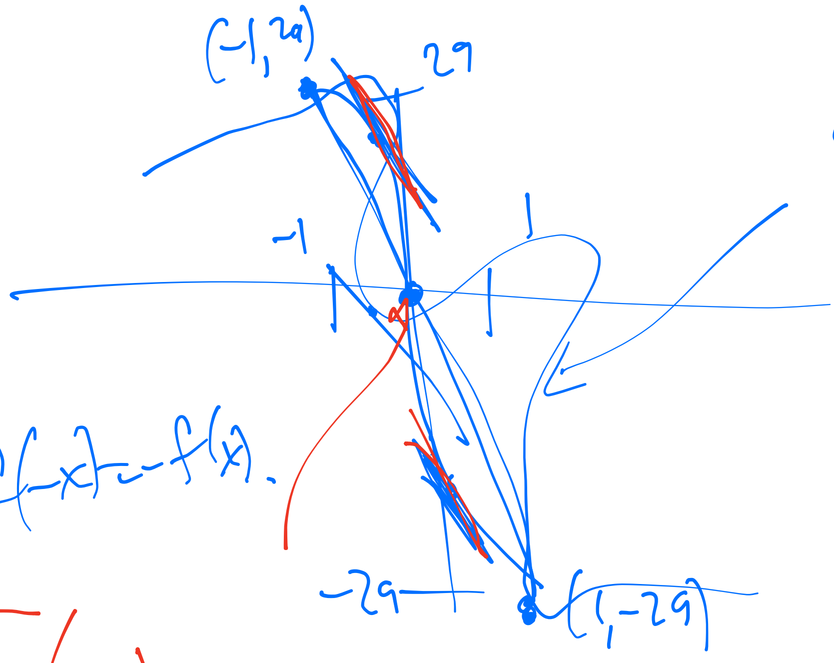
(b)

$$4 + \frac{\overset{4}{\cancel{f(2)}} - \cancel{f(1)}}{\cancel{2-1}} > -4 + 4 + f(1)$$

$$\boxed{8 > f(1)}$$

6. Suppose f is an odd function that is differentiable for all real numbers. Fill in the blanks.

If $f(1) = -29$, then the Mean Value Theorem says there is/are at least $\boxed{\cancel{1}\ 2}$ tangent line(s) to the graph of $y = f(x)$ with slope $m = \boxed{-29}$.

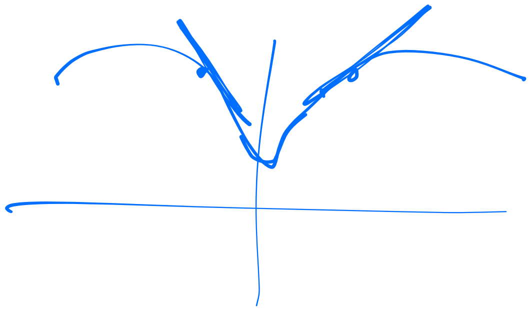


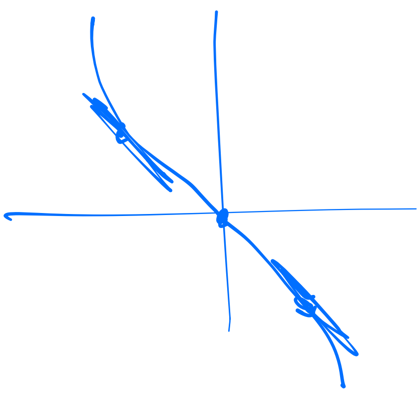
avg slope: $\dfrac{-29 \cdot 2}{2} = -29$

$f(-x) = -f(x)$.

$f(0) = 0$

If f even:





$f(x) = -f(x)$

$f(-0) = -f(0)$

$f(0) = -f(0) \Rightarrow f(0) = 0.$

7. Let $f(x) = x^2e^{-7x}$. Use f in parts (a)-(f).

(a) Find $f'(x)$.

(b) Find the critical points of f.

(c) Find the open intervals where f is increasing.

(d) Find the open intervals where f is decreasing.

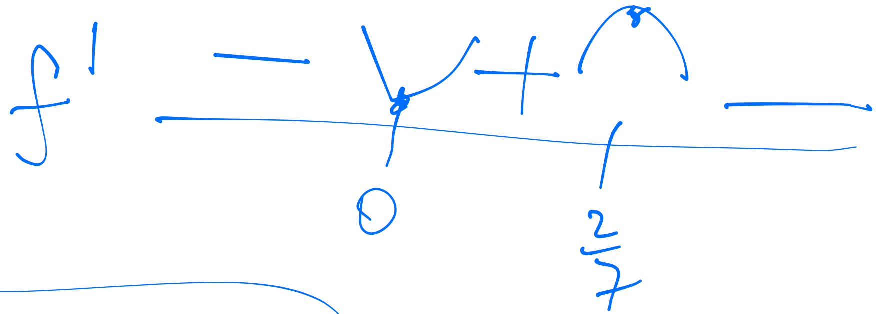
(e) Find the local minima and their locations.

(f) Find the local maxima and their locations.

a) $f'(x) = 2xe^{-7x} + x^2e^{-7x}(-7)$

$f'(x) = e^{-7x} \cdot x\left[2 - 7x\right].$

b) Crit values: $x = 0, \frac{2}{7}$.



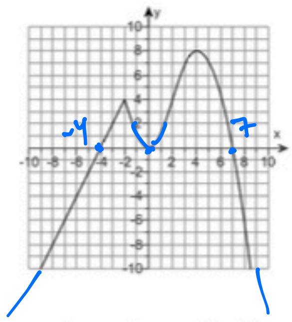
c) $(0, \frac{2}{7})$

d) $(-\infty, 0), \ (\frac{2}{7}, \infty)$

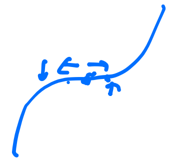
e) Local min at $x = 0$, $f(0) = 0$.

f) Local max at $x = \frac{2}{7}$, $f(\frac{2}{7}) = (\frac{2}{7})^2 e^{-2}$

8. The graph of $y = f'(x)$ – **the derivative of** f is sketched below.



$y = f'$



Use the graph of $y = f'(x)$ to answer questions about the ***continuous function*** f in parts (a)-(g). Again, the graph depicted above is for $y = f'(x)$, *not* $y = f(x)$.

(a) Find the interval(s) where f is increasing. $\to [-4, 7]$

(b) Find the interval(s) where f is decreasing. $\to (-\infty, -4), (7, \infty)$

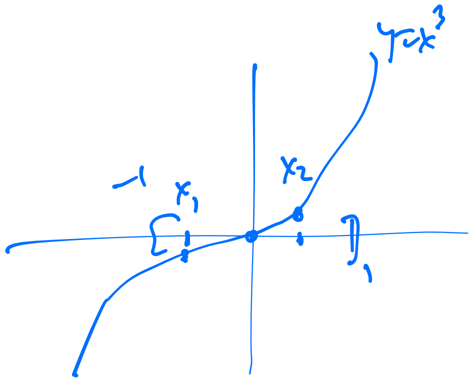
(c) Find the interval(s) where f is concave up. (f' incr) $\to (-\infty, -2), (0, 4)$

(d) Find the interval(s) where f is concave down. $\to (-2, 0), (4, \infty)$

(e) Find the critical points of f. $-4, 0, 7$

(f) Find the location(s) of any local maxima of f. $\to$ + − loc max $x = 7$

(g) Find the location(s) of any local minima of f. $\to$ − + at $x = -4$



Def: f is (strictly) increasing on $[a, b]$

if: $\forall\ a \le x_1 < x_2 \le b$,

$f(x_1) < f(x_2)$.

9. Suppose $f(x)$ is a function whose derivative is

$$f'(x) = x^2(16 - x). = 16x^2 - x^3$$

$$f(x) = 16\frac{x^3}{3} - \frac{x^4}{4} + C.$$

Use this information in parts (a)-(g).

(a) Find the domain of f. $\mathbb{R}$, $(-\infty, \infty)$.

(b) Find the critical points of f. $x = 16, 0,$

(c) Find the interval(s) where f is increasing and where f is decreasing. Use interval notation to describe the intervals and clearly label your answers.

(d) Find the x-values where f has local extrema. Be sure to identify the type of local extrema you find.

(e) Find the intervals of concavity for f. Use interval notation to describe the intervals of concavity and clearly state the type of concavity for each of your intervals.

(f) Find the x-values where f has inflection points.

10. Compute the limit. If a limit does not exist, state why this is the case. If you use L'Hôpital's Rule, be sure to indicate where it is being applied and the indeterminate form that allows its use. Remember, your use of limit notation will be graded.

(a) $\lim_{x\to 0} \frac{\sin(8x) - 8x}{14x^3}$

(b) $\lim_{x\to 6^+} \left(\frac{1}{x-6} - \frac{1}{\ln(x-5)}\right)$

(c) $\lim_{x\to\infty} \left(\frac{x+8}{x-8}\right)^{7x}$

(d) $\lim_{x\to 0} \frac{f(\tan(6x))}{9x + f(\sin x)}$, if f is a differentiable function such that $\lim_{x\to 0} f(x) = 0$ and $\lim_{x\to 0} f'(x) = 13$.

(a) $\lim_{x\to 0} \frac{\sin(8x) - 8x \to 0}{14x^3 \to 0}$ Indet $\frac{0}{0}$ Form.

$\overset{L'H}{=} \lim_{x\to 0} \frac{\cos(8x)\cdot 8 - 8 \to 0}{14\cdot 3\cdot x^2 \to 0}$ Indet $\frac{0}{0}$ Form

$\overset{L'H}{=} \lim_{x\to 0} \frac{-\sin(8x)\cdot 8\cdot 8 \to 0}{14\cdot 3\cdot 2x \to 0}$ Indet $\frac{0}{0}$.

$\overset{L'H}{=} \lim_{x\to 0} \frac{-8^2\cdot \overset{1}{\cancel{\cos(8x)}}\cdot 8}{14\cdot 3\cdot 2} = \boxed{\frac{-8^3}{14\cdot 3\cdot 2}}$

b) $\lim_{x\to 6^+}\left(\frac{1}{x-6} - \frac{1}{\ln(x-5)}\right) = \lim_{x\to 6^+} \frac{\ln(x-5) - (x-6) \to 0}{(x-6)\ln(x-5) \to 0}$ Indet $\frac{0}{0}$ Form

$\overset{L'H}{=}$ (l'H) $\lim_{x\to 6^+} \frac{\frac{1}{x-5} - 1 \to 0}{1\cdot \ln(x-5) + \frac{x-6}{x-5} \to 0}$ Indet $\frac{0}{0}$ Form

mult top & bot by $(x-5)$. ✩

$$= \lim_{x\to 6^+} \frac{1-(x-5) \to 0}{(x-5)\ln(x-5)+x-6 \to 0}$$

$$\overset{?}{=} \lim_{x\to 6^+} \frac{-1}{\frac{(x-5)}{x-5} + 1\cdot \underbrace{\ln(x-5)}_{\to 0} + 1} = \boxed{-\frac{1}{2}.}$$

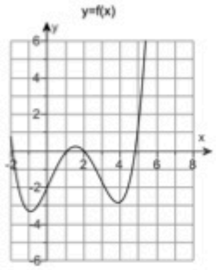
(d)

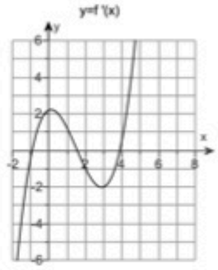
$$\lim_{x\to 0} \frac{f(\tan(6x)) \to 0}{9x+f(\sin x) \to 0} \quad \boxed{\begin{array}{l} f(0)=0 \\ f'(0)=13 \end{array}} \quad \overset{?}{=} \lim_{x\to 0} \frac{\overbrace{f'(\tan(6x))}^{\to 13}\cdot \sec^2(6x)^{\to 1}\cdot 6}{9+\underbrace{f'(\sin x)}_{\to 13}\cdot \underbrace{\cos x}_{\to 1}}$$

$$= \boxed{\frac{13\cdot 6}{9+13}}$$

---

11. Portions of the graphs of $y = f(x)$ and $y = f'(x)$ are sketched below:





Use the information in these graphs to evaluate each of the following limits, if possible. If not possible, explain why not.

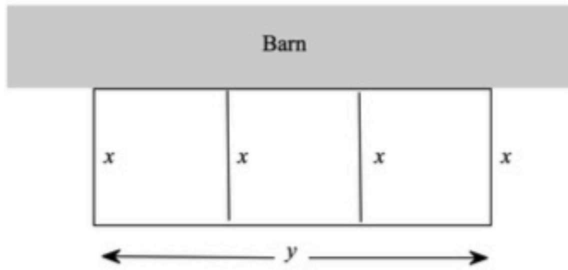
(a) $\displaystyle\lim_{x\to 2} \frac{2 \cdot f(x)}{3x^2 - 18x + 24}$

(b) $\displaystyle\lim_{x\to 2} \frac{2 + f(x)}{3x^2 - 18x + 24}$

(c) $\displaystyle\lim_{x\to 2} \left(2 + \frac{f(x)}{3x^2 - 18x + 24}\right)$

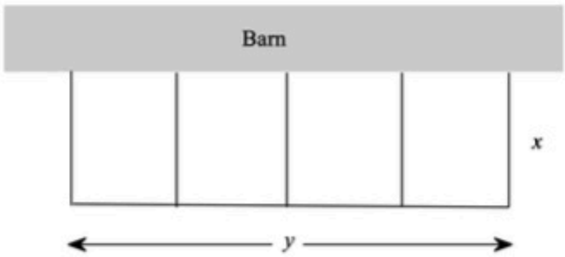
(d) $\displaystyle\lim_{x\to 0} \frac{f(2 + x)}{3x^2 - 18x + 24}$

12. Farmer Green has 600 feet of fencing to build a rectangular pen, which will be divided into 3 identical and adjacent pens. Farmer Green decides that the barn will form one side of the pen, and so, will not require fencing. Three exterior and two interior fences partition the pen into three congruent, smaller rectangular pens as depicted in the figure below:



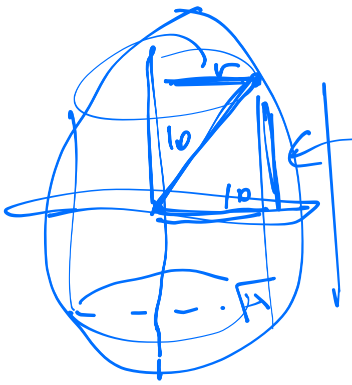
What dimensions for the (outer) rectangular pen maximize the enclosed area?

13. Farmer Green needs yet more pens for her animals. She has another very large barn that she wishes to build four identical and adjacent rectangular pens against. She will use the side of the barn to enclose one side of the pens, and so, will not need to use fencing along the barn wall. Each of these pens will have an area of 100 ft^2, with the four pens enclosing a total area of 400 ft^2. A sketch of her plan is seen below:



Find the minimum amount of fencing to build the four pens in the manner indicated.

14. Find the dimensions of a right circular cylinder of maximum volume that can be inscribed in a sphere of radius 10 cm. What is the maximum volume?



$$V = \pi r^2 h.$$

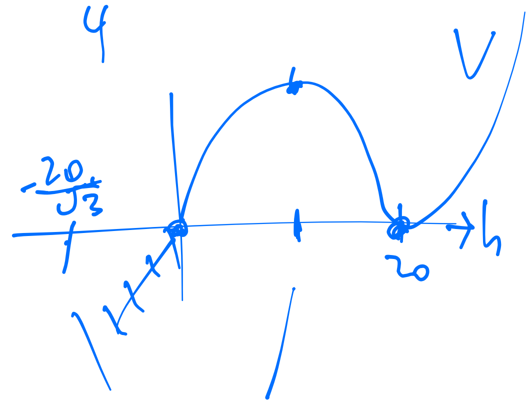
cylinder inscribed in sphere

$$r^2 + \left(\frac{h}{2}\right)^2 = 100$$

$$r^2 = 100 - \frac{h^2}{4}$$

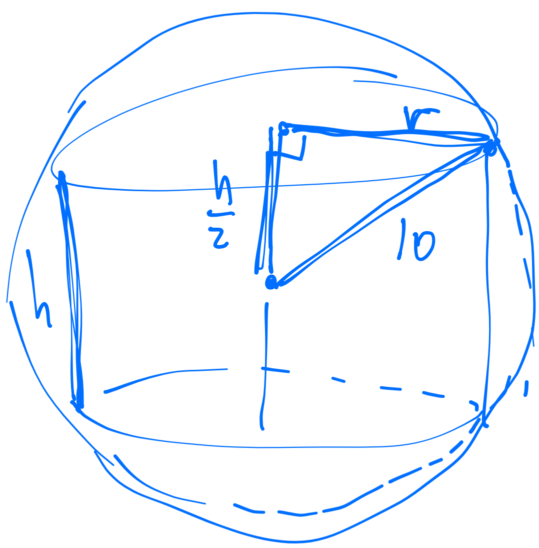
$$V = \pi\left(100 - \frac{h^2}{4}\right)h$$

$$= 100\pi h - \frac{\pi}{4}h^3.$$

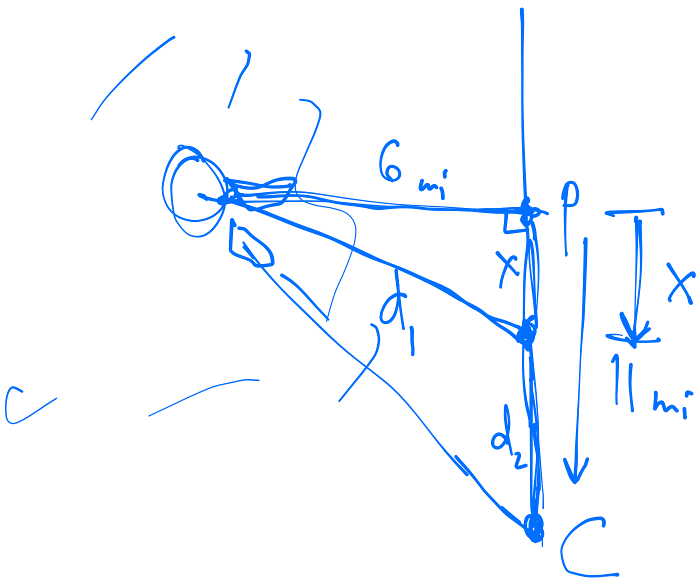


$$V' = 100\pi - \frac{3\pi}{4}h^2 = 0 \Rightarrow h = \pm\sqrt{\frac{400}{3}} = \boxed{\frac{1}{\sqrt{3}} \cdot 20}$$

$$V\left(\frac{20}{\sqrt{3}}\right) = \boxed{\pi\left(100 - \frac{400}{3 \cdot 4}\right) \cdot \frac{20}{\sqrt{3}}}$$



15. A small island is 6 miles from the nearest point P on the straight shoreline of a large lake. If a woman on the island can row a boat 3 miles per hour and can walk 4 miles per hour, where should the boat be landed in order to arrive at a town 11 miles down the shore from P in the least time?



$d_1 = \sqrt{x^2+6^2}$

$d_2 = 11-x$.

x = where she lands.

$$T = \frac{\sqrt{x^2+6^2}}{3} + \frac{11-x \text{ mi}}{4 \text{ mi/hr}} \text{ hr}$$

16. Find the point on the line $\frac{x}{7} + \frac{y}{4} = 1$ closest to the origin. What is the distance between this point and the origin? (Hint: it is easier to optimize the ***square*** of the distance.)

17. Solve the initial value problem:

$$\frac{dv}{dt} = \frac{4}{1+t^2} + \sec^2 t, \quad v(0) = 2.$$

$$v' = 4 \cdot \frac{1}{1+t^2} + \sec^2 t.$$

$$v(t) = 4 \cdot \tan^{-1}(t) + \tan t + C.$$

$$v(0) = \cancel{4 \tan^{-1}(0)} + \cancel{\tan 0} + C = 2$$

$$\boxed{v(t) = 4\tan^{-1} t + \tan t + 2}$$

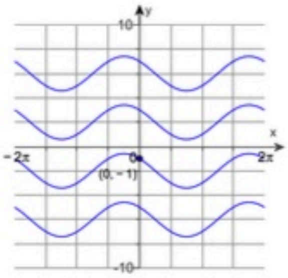
18. (a) Find a curve $y = f(x)$ with the following properties:

 (i) $\dfrac{d^2y}{dx^2} = 6x$

 (ii) Its graph passes through the point $(0, 1)$, and has a horizontal tangent line there.

(b) How many curves like this are there? How do you know?

19. The graph below shows solution curves of the differential equation $\frac{dy}{dx} = -\sin x - \cos x$.



Find an equation for the curve that passes through the labeled point.

20. A car is moving at 60 mi/hr (88 ft/sec) on a very long and straight Midwestern highway when the driver begins to accelerate at 21 ft/sec^2 for 15 seconds. How far did the car travel during the 15 seconds of acceleration?