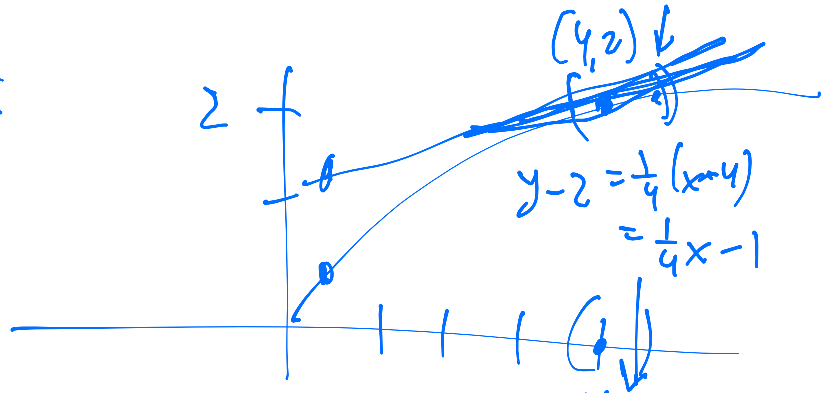
Recall: Product Rule, Quotient Rule, Chain Rule, Implicit Diff, Related Rates.

New: Linearization.

Eg.: What is $\sqrt{4.01}$? "Zeroth order" approximation: $\sqrt{4} = 2$.

$L(4.01)$ $y = L(x)$ = tangent line.

$f(x) = \sqrt{x}$



$y - 2 = \frac{1}{4}(x-4)$
$= \frac{1}{4}x - 1$

$y' = \frac{1}{2}x^{-1/2}\Big|_{x=4} = \frac{1}{2\cdot 2} = \frac{1}{4}$.

Idea: Tangent line to $y=\sqrt{x}$ at $(4,2)$ is a "first order" = "linear" approximation to $f(x)=\sqrt{x}$ in a "neighborhood" of $x=4$.

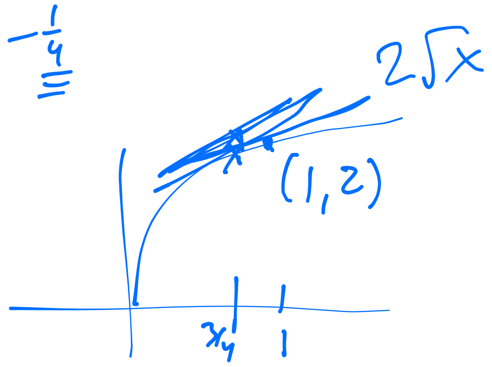
$\boxed{L(x) = \frac{1}{4}x + 1}$ So approximate $f(4.01)$ by $L(4.01)$

$L(4.01) = 1.0025 + 1 = 2.0025$. Real: $\sqrt{4.01} = 2.002498$.

Ex.: $\sqrt{3}$? $\sqrt{3} = \sqrt{4-1} = 2\sqrt{1-\frac{1}{4}}$

$f(x) = 2\sqrt{x}$, near $x = 1$

$f'(x)\Big|_{x=1} = 2\cdot\frac{1}{2}x^{-1/2} = \frac{1}{\sqrt{x}}\Big|_{x=1} = 1$



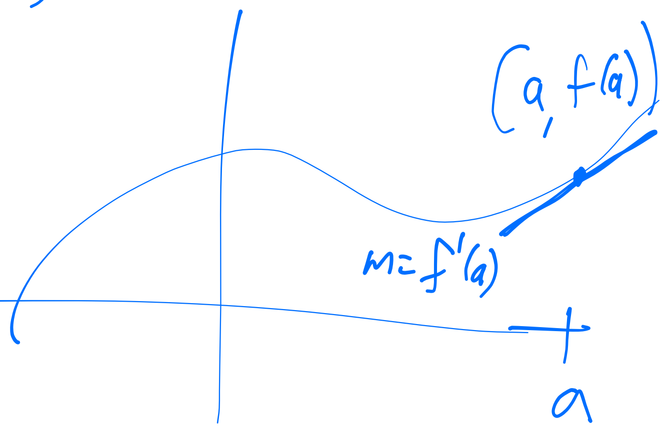
Linear approximation: $L(x) = 2 + 1(x-1)$

$L(\frac{3}{4}) = 2 + (-\frac{1}{4}) = 1.75$ $\qquad \sqrt{3} = 1.732\ldots$

General procedure: $y = f(x)$

Linear approximation to $f(x)$ near $x = a$ is

$$\boxed{L(x) = f(a) + f'(a)(x-a)}$$



$$y - f(a) = f'(a)(x-a)$$

Quiz: What is the Linear Approximation of $\sqrt[3]{7}$?

$f(x) = \sqrt[3]{x} = x^{1/3}$, $\quad a = 8$ ("anchor").

$f(8) = 8^{1/3} = 2$, $\quad f'(x) = \frac{1}{3}x^{-2/3}$, $\quad f'(8) = \frac{1}{3 \cdot 8^{2/3}} = \frac{1}{3 \cdot 4} = \frac{1}{12}$.

$L(x) = 2 + \frac{1}{12}(x-8)$.

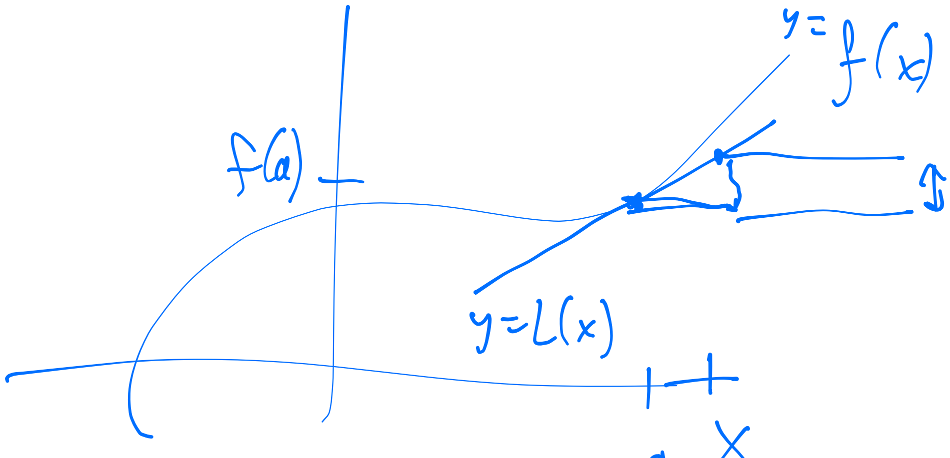
is the linear approx to $f(x) = x^{1/3}$ near $x = 8$.

Eval at $x = 7$. Then $\sqrt[3]{7} \approx 2 + \frac{1}{12}(7-8) = \frac{23}{12}$.

$\frac{23}{12} = 1.9166\ldots$, $\quad \sqrt[3]{7} = 1.9129\ldots$

$$\boxed{L(x) = f(a) + \underbrace{f'(a)(x-a)}}$$

What's really going on?



$f'(a)(x-a) = dy$.
differential in y.

$\underbrace{a \quad x}$

dx = "little change in x"

For example, $f(x) = x^{1/3}$, near $x = 8 = a$.

$$L(x) = \underbrace{2} + \underbrace{\frac{1}{12}(x-8)}$$

change from "known" value

$$dy = f'(x)\cdot dx, \qquad \frac{dy}{dx} = f'(x).$$

→ used to mean: infinitesimal change in x gives rise to infinitesimal change in y, which in the limit measures the derivative.

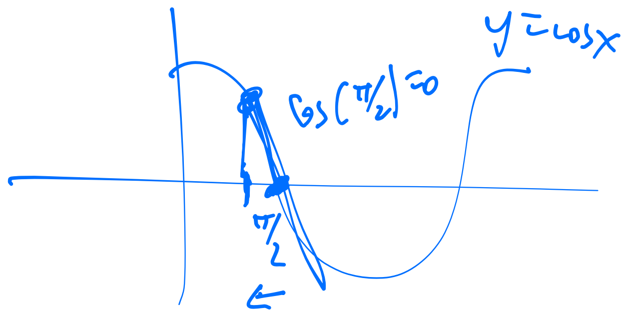
Now, don't take the limit as $dx \to 0$, just leave dx finite ("small" value), will get small finite value for dy.

↳ "Differential" $\Delta y = f'(x)\cdot \Delta x$

Eg: What is differential of $y=\cos x$,

$dy = \underline{-\sin x}\ \underline{dx}$ (in implicit differentiation, if $x=x(t)$, $y=y(t)$, then $\frac{dy}{\cancel{dt}} = -\sin x\, \frac{dx}{\cancel{dt}}$).

Eg: Approximate $\cos\left(\frac{\pi}{2}-0.1\right)$.



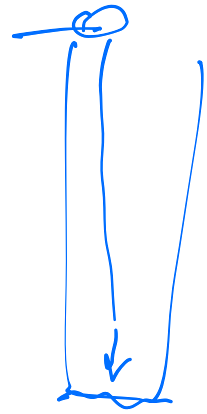
$y=\cos x$, $\underline{dx} = \underline{-0.1}$

$dy = -\sin x\Big|_{x=\pi/2}(-0.1) = -1(-0.1) = 0.1$

Differential dy tells you by how much y changes when x changes by dx.

Eg: $d(\tan x) = \sec^2 x\cdot dx$,

Application: Deep well, measure depth by dropping stone, waiting for splash.



$s(t) = 0 + 0t - 32\frac{ft}{s^2}\cdot\frac{t^2}{2} = -16t^2$.

Error estimate for stop watch of 0.1 sec.

What is the estimate of the error in measuring depth?

$$\boxed{ds = -32t \cdot dt} = -32t(0.1) = \boxed{-3.2t}$$

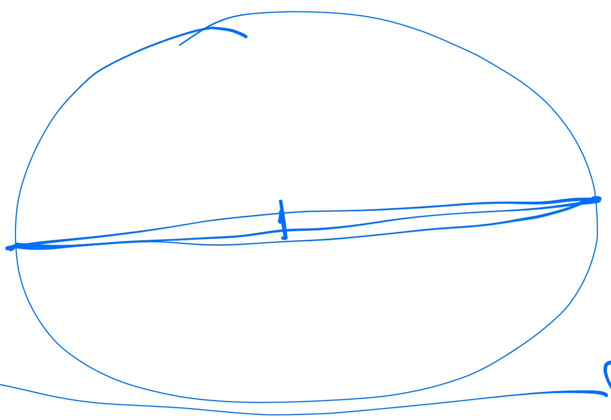
If $t = 2$, $s = 64$ ft, $ds = -6.4$,

If $t = 5$, $s = 400$ ft, $ds = -16$.

true depth is likely 384 ft, not 400..

E.g.: Big circular field, need to know its area. Walk length of diameter, and estimate from there.

$$A = \pi r^2$$

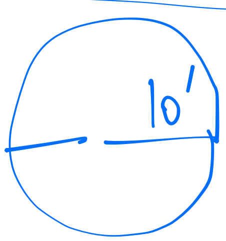


$2r$

Shoes are 13", not 12"

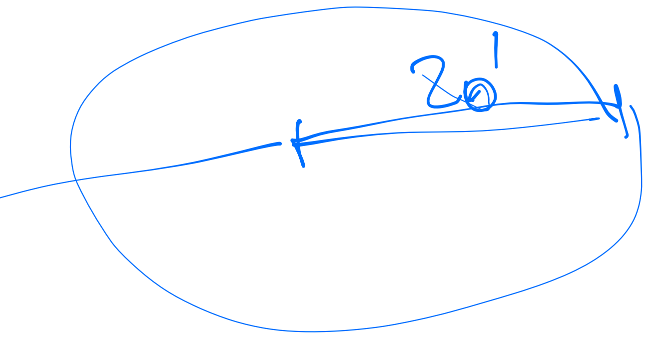
$dr = -1"$,

$$dA = 2\pi r \cdot dr$$
$$\approx 6 \cdot (r)(-1)$$



$A = \pi \cdot 100$
$= 314$

$dA = -60\ \text{ft}^2$.



$A = 400\pi$
$= \boxed{1256}$

$dA = -6 \cdot 20$
$= -120$

More accurate estimate of area: 1136 in^2.

$$d(x^5 + 7x) = (5x^4 + 7)\,dx.$$

Ex: 

Wake up at 8 am, get out of bed, hit snooze,

Time out of bed once awake is $f(t) = t^2$.

$$df = 2t \cdot dt.$$

Hit snooze at 8:20, $dt = 3$ s. $= \frac{1}{20}$ min.

Expected Time out of bed $=$ 400 min after 8am.

Differential: $df = 2 \cdot 20 \cdot \frac{1}{20} = 2$ min.