# Review Sheet for Math 151 Midterm 2 Fall 2022

The following questions are intended to give you practice working problems that test the ideas covered in the course for the second midterm exam. The number of problems on this review sheet is larger than the number of problems on a typical 80-minute exam – this sheet is NOT A PRACTICE TEST. You should not memorize the problems, as the problems you encounter on the midterm exam will not look exactly like the problems on this review sheet – the exam problems will be different. Your goal is to be able to think through and understand the processes required to answer the questions correctly. Before you work the review problems, you should study for the exam and when you feel you have prepared enough, try doing the problems on this sheet WITHOUT looking at your notes, textbook, or videos. Make sure to try this a couple of days before the midterm so that you will have time to fill in any gaps of knowledge you uncover. If you start your studying by doing the review sheet first, you will not maximize the benefit of the review sheet.

1. Use the definition of derivative to find $f'(2)$ for $f(x) = \dfrac{3}{x^2+4}$. $=3(x^2+4)^{-1}$

$$\lim_{u\to 2}\frac{\frac{3}{u^2+4}-\frac{3}{2^2+4}}{u-2} = \lim_{u\to 2}\frac{3\cdot 8-3(u^2+4)}{(u^2+4)\cdot 8\cdot(u-2)} = \lim_{u\to 2}\frac{-3\overbrace{(u^2-4)}^{(u+2)\cancel{(u-2)}}}{(u^2+4)8\cancel{(u-2)}}$$

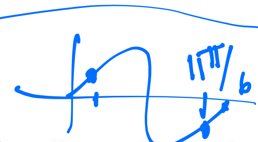
$$= \boxed{\frac{-3\cdot(2+2)}{8\cdot 8}} \stackrel{?}{=} \boxed{\text{Check: } -3(x^2+4)^{-2}(2x)\Big|_{x=2}}$$

2. Use the definition of derivative to compute $f'(x)$ for $f(x) = 4x^2 - 5x + 1$.

$$\lim_{u\to x}\frac{4u^2-5u+\cancel{1}-(4x^2-5x+\cancel{1})}{u-x} = \lim_{u\to x}\frac{4\overbrace{(u^2-x^2)}^{(u+x)(u-x)}-5\cancel{(u-x)}}{\cancel{u-x}} = 4(x+x)-5$$

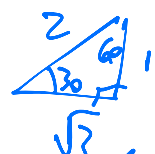
$$\lim_{h\to 0}\frac{4(x+h)^2-5(x+h)+1-(4x^2-5x+1)}{h}$$

Check: $8x-5$



3. Each of the limits below can be interpreted as a derivative $f'(a)$. For each limit, determine $f(x)$ and $a$, and then use the rules of differentiation to compute the limit by finding $f'(a)$.

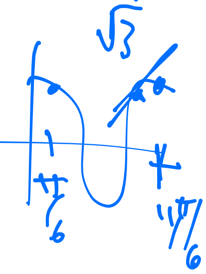
(a) $\displaystyle\lim_{x\to 2}\frac{x^{2022}-2^{2022}}{x-2}$ $= \boxed{2022\cdot 2^{2021}}$ (b) $\displaystyle\lim_{h\to 0}\frac{\ln(15+h)-\ln(15)}{h}$ (c) $\displaystyle\lim_{x\to 11\pi/6}\frac{\cos x-\sqrt{3}/2}{x-11\pi/6}$



$= f'(a)$ when $f(x) = x^{2022}$ & $a=2$, $f'(x) = 2022x^{2021}\Big|_{x=2}$

$f(x) = \ln x$, $a = 15$, $f'(x) = \frac{1}{x}\Big|_{x=a} = \frac{1}{15}\Big|_{x=15}$

$f(x) = \cos x$, $a = 11\pi/6$, $f'(x) = -\sin x\Big|_{x=11\pi/6}\Big|_{x=11\pi/6} = +\frac{1}{2}$



4. Compute the derivative.

(a) $f(x) = 2x^3 e^{-4x}\cos(5x)$

(b) $h(t) = \dfrac{1+\tan(5t)}{1-\tan(5t)}$

(c) $f(x) = \sqrt{1+\sqrt{1+\sqrt{x}}}$

(d) $y = \sec^{-1}(te^{3t})$

(e) $y = \sin(\csc(3e^{-4x}))$

(f) $y = x\tan^{-1}\sqrt{x}$

(g) $y = \ln(x\log_2 x)$

(h) $y = \dfrac{\sqrt{x+1}(2-x)^5}{(x+3)^7}$ log-diff

(i) $g(z) = (e^{3z}-e^{-3z})^2(e^{3z}+e^{-3z})$

(j) $y = (\ln x)^{\cos(3x)}$

(a) $\ln y = \ln 2 + 3\ln x - 4x + \ln\cos(5x)$

$\frac{1}{y}\cdot y' \quad y' = (2x^3 e^{-4x}\cos(5x))\left[\frac{3}{x} - 4 + \frac{1}{\cos 5x}\cdot(-\sin(5x))\cdot 5\right]$

(e) $y' = \cos(\csc(3e^{-4x}))\cdot\left[-\csc(3e^{-4x})\cot(3e^{-4x})\right]\cdot 3\cdot e^{-4x}(-4)$

(c) $y = \left[1+(1+x^{1/2})^{1/2}\right]^{1/2}$

$y' = \frac{1}{2}\left(1+(1+x^{1/2})^{1/2}\right)^{-1/2}\cdot\left(\frac{1}{2}(1+x^{1/2})^{-1/2}\right)\cdot\frac{1}{2}x^{-1/2}$

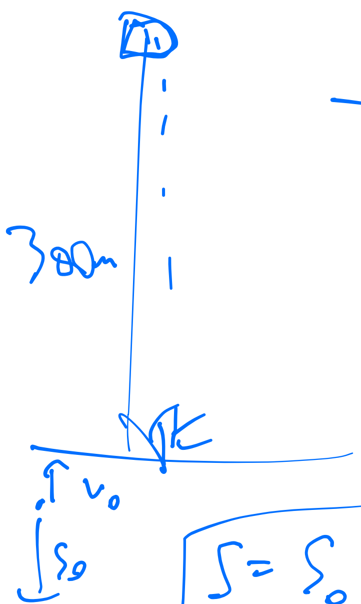
(f) $y = x\cdot\tan^{-1}(\sqrt{x})$, $y' = 1\cdot\tan^{-1}(\sqrt{x}) + x\cdot\left[\frac{1}{1+(\sqrt{x})^2}\right]\cdot\frac{1}{2}x^{-1/2}$

5. Find a formula for $\frac{d^n}{dx^n}[e^{-2x}]$ where $n \geq 0$ is an integer.

$$\frac{d}{dx}\left[e^{-2x}\right] = e^{-2x}(-2). \qquad \frac{d^2}{dx^2}\left[e^{-2x}\right] = e^{-2x}(-2)(-2).$$

$$\frac{d^n}{dx^n}\left[e^{-2x}\right] = e^{-2x}(-2)^n.$$

6. Find the velocity of an air conditioner accidentally dropped from a height of 300 m at the moment it hits the ground.




$$\boxed{a(t) = -9.8\,\frac{m}{s^2}}$$

$$S = 300m + 0t - \frac{9.8}{2}t^2\,\frac{m}{s^2}.$$

$$V\Big|_{t=t_0} = \frac{ds}{dt} = -9.8t\Big|_{t=t_0} = -9.8\cdot\sqrt{\frac{300\cdot 2}{9.8}}\ \frac{m}{s}.$$

at moment of time: $S = 0, = 300 - \frac{9.8}{2}t^2\,\frac{m}{s^2}$

$$t_0 = \sqrt{\frac{300\cdot 2}{9.8}}\ s.$$

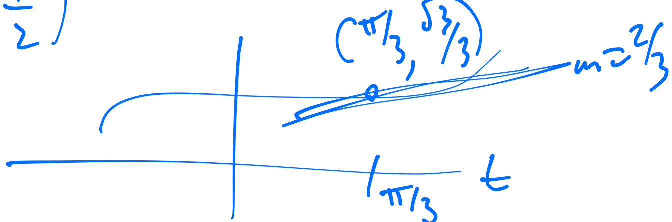
$$\boxed{S = S_0 + V_0 t - \frac{9.8}{2}t^2\,\frac{m}{s^2}.}$$

7. Find an equation of the tangent line to to the graph of $y = \frac{\sin t}{1+\cos t}$ when $t = \frac{\pi}{3}$.

$$y'\Big|_{t=\pi/3} = \frac{(1+\cos t)\cdot\cos t - \sin t(-\sin t)}{(1+\cos t)^2}$$


$$= \frac{\left(1+\frac{1}{2}\right)\frac{1}{2} + \left(\frac{\sqrt{3}}{2}\right)^2}{\left(1+\frac{1}{2}\right)^2} = \frac{3}{2}\cdot\frac{4}{9} = \frac{2}{3}.$$

$$y(t) = \frac{\sqrt{3}}{2}\cdot\frac{2}{3} = \frac{\sqrt{3}}{3}.$$




$$\boxed{y - \frac{\sqrt{3}}{3} = \frac{2}{3}\left(t - \frac{\pi}{3}\right)}$$

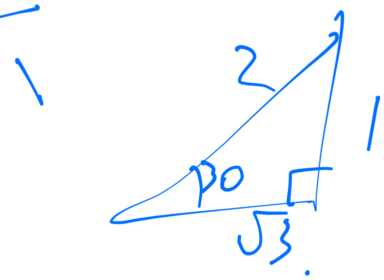
$y=$

8. Compute the derivative of $h(\sin x)$ at $x = \dfrac{\pi}{6}$, assuming that $h'(0.5) = 10$.

$$y' = h'(\sin x)\cdot \cos x \Big|_{x=\pi/6}$$

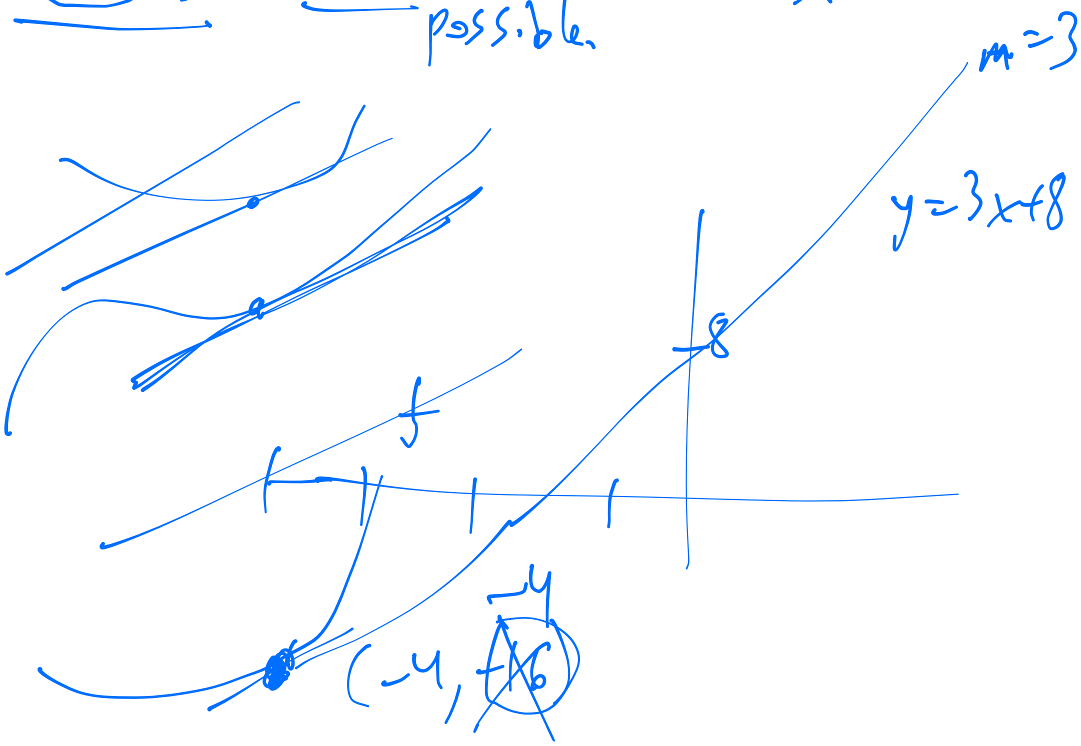
$$= h'\left(\tfrac{1}{2}\right)\cdot \frac{\sqrt{3}}{2}$$

$$= 10 \cdot \frac{\sqrt{3}}{2}.$$




9. Let $f(x)$ and $g(x)$ be functions so that $f'(-4) = g'(-4)$ and the line tangent to $f(x)$ at $x = -4$ is $y = 3x + 8$. Compute the following values, if possible.

(a) $f(-4) = -\cancel{16}$ $-4$ (b) $f'(-4) = 3$ (c) $g(-4)$ Not possible. (d) $g'(-4) = 3$.



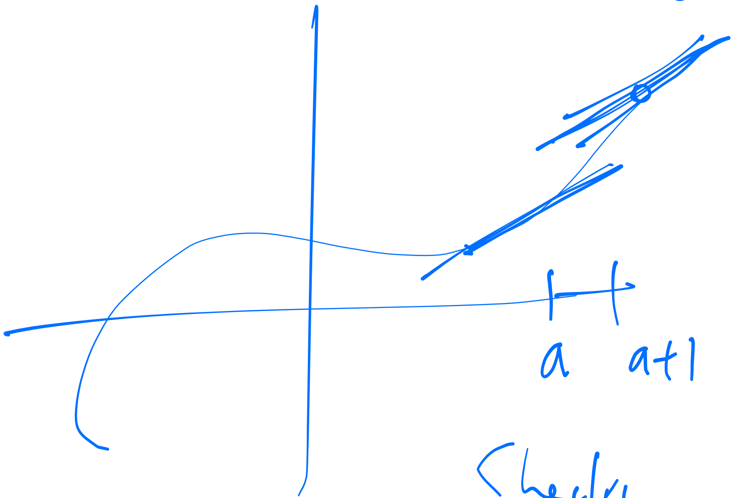
10. Find a number $a$ so that the tangent lines to $y = x^3 + 11x^2 + x + 7$ at $x = a$ and at $x = a + 1$ are parallel.

$y' = 3x^2 + 22x + 1$

Need:

$3a^2 + \cancel{22a} + \cancel{1} = 3(a+1)^2 + 22(\cancel{a}+1) + \cancel{1}$

$\cancel{3a^2} = \cancel{3a^2} + 6a + 3 + 22$



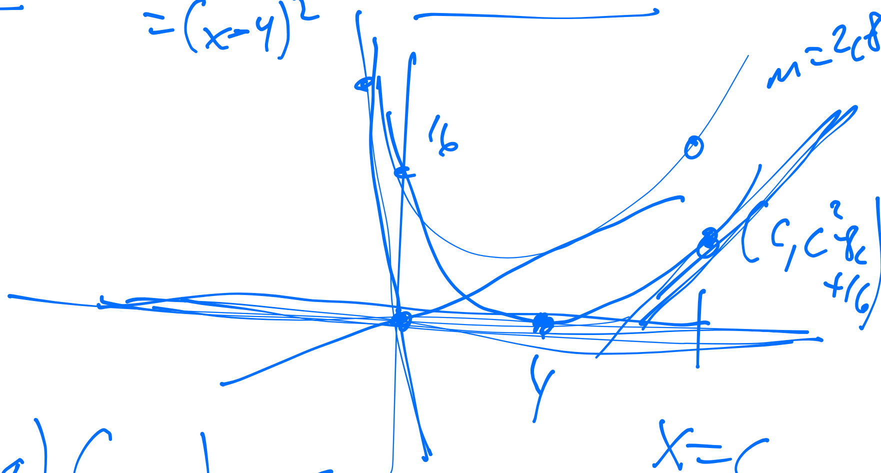

$a = -\frac{25}{6}$.

Check,

11. Find the equations of all tangent lines to the graph of $y = x^2 - 8x + 16$ that pass through the origin $(0, 0)$.

$= (x-4)^2$

$y'\big|_{x=c} = 2c - 8$




Eqn. of tangent line:

$y - (c^2 - 8c + 16) = (2c-8)(x-c)$.

$0 = \cancel{c^2} - \cancel{8c} + 16 - \cancel{2}c^2 + \cancel{8c}$

$c = 4, -4$

$y - (\cancel{4^2 - 8 \cdot 4 + 16}) = (\cancel{2 \cdot 4 - 8})(x-4) \iff y = 0.$

& $y - ((-4)^2 - 8(-4) + 16) = (2(-4) - 8)(x - (-4))$.

12. Suppose that $f(x) = \begin{cases} 6x^2+7, & \text{if } x \le -2 \\ ax+b, & \text{if } x > -2 \end{cases}$; where $a$ and $b$ are some constants. Use $f(x)$ in parts (a)-(e).

(a) Compute $\lim_{x \to -2^-} f(x)$ $= 31$

(b) Compute $\lim_{x \to -2^+} f(x)$ $= -2a+b$ !

$\boxed{-2a+b=31}$

(c) If $f(x)$ is to be continuous at $x = -2$, what equation involving $a$ and $b$ must be true?

(d) Assuming $f$ is continuous at $x = -2$, compute $f'_-(-2) = \lim_{h \to 0^-} \dfrac{f(-2+h) - f(-2)}{h}$. $= 12(-2) = -24$

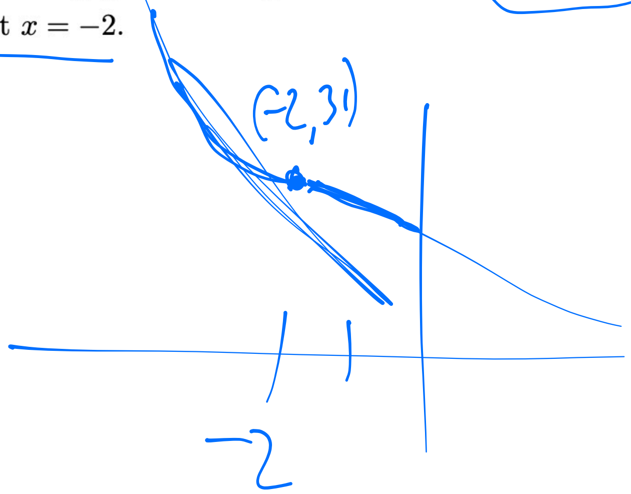
(e) Assuming $f$ is continuous at $x = -2$, compute $f'_+(-2) = \lim_{h \to 0^+} \dfrac{f(-2+h) - f(-2)}{h}$. $= a.$

(f) Find values for $a$ and $b$ that make $f$ differentiable at $x = -2$.

$\boxed{a = -24, \quad b = -17}$

$-2a+b=31$

$48+b=31,$



13. The function $s = f(t) = t^3 - 6t^2 + 10t$, $0 \le t \le 4$, gives the position of an object moving along the $s$ axis as a function of time $t$ (in sec). Assume the motion is horizontal. Use $s$ in parts (a)-(g).

(a) Compute $v(t)$ and $a(t)$. $v(t) = 3t^2 - 12t + 10$, $a(t) = 6t - 12$.

(b) When is the object momentarily at rest? $t = \left(12 \pm \sqrt{12^2 - 4(3)(10)}\right)/(2\cdot 3) = t_+, t_-$

(c) When does the object move to the left? When does the object move to the right? $\in (t_-, t_+)$ $[0, t_-) \cup (t_+, 4]$

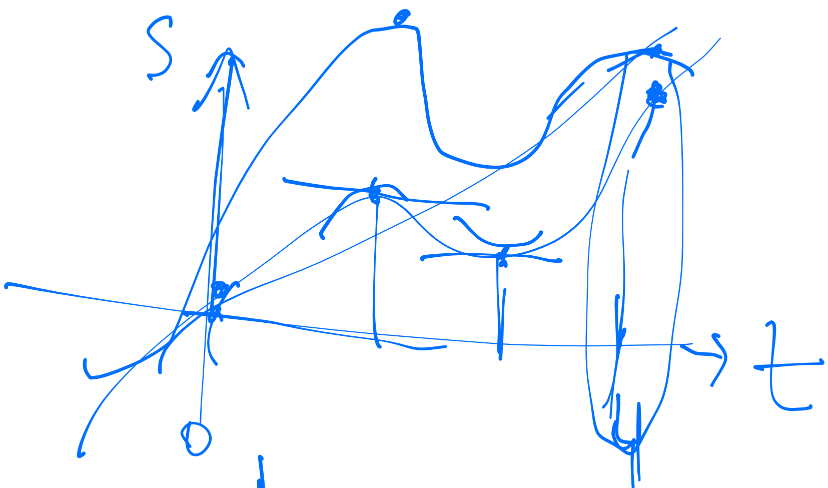
(d) When does the object change direction? at $t_+$ & $t_-$

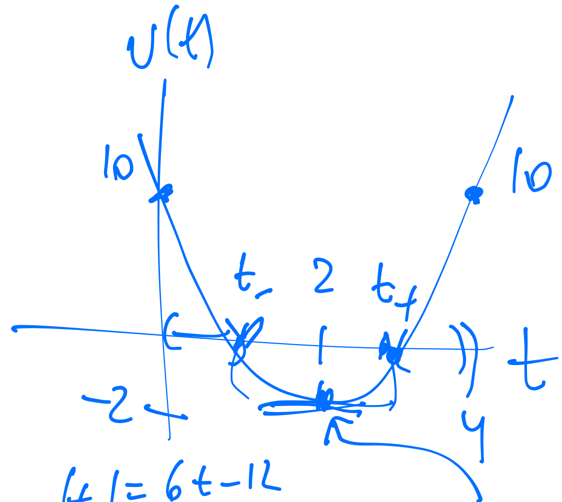
(e) When does the object speed up and slow down? $(2, 4]$, $[0, 2)$

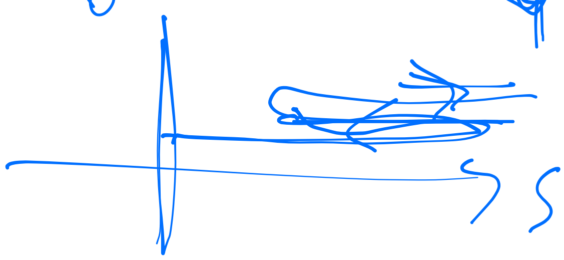
(f) When is the object moving fastest (highest speed)? Slowest? $t = 2$.

$t = 0$ & $= 4$

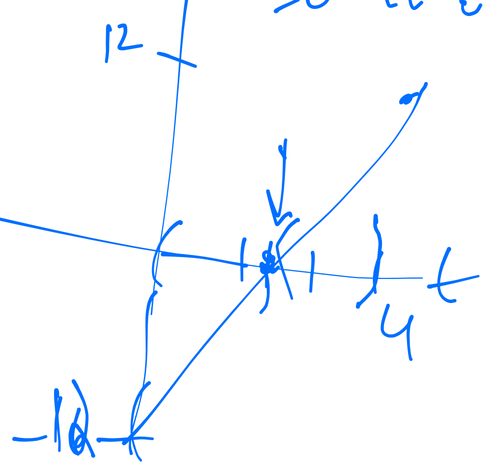
~~(g)~~ When is the object farthest from the axis origin? What is this farthest distance?




$a(t) = 6t - 12$
$= 0$ at $t = 2$.

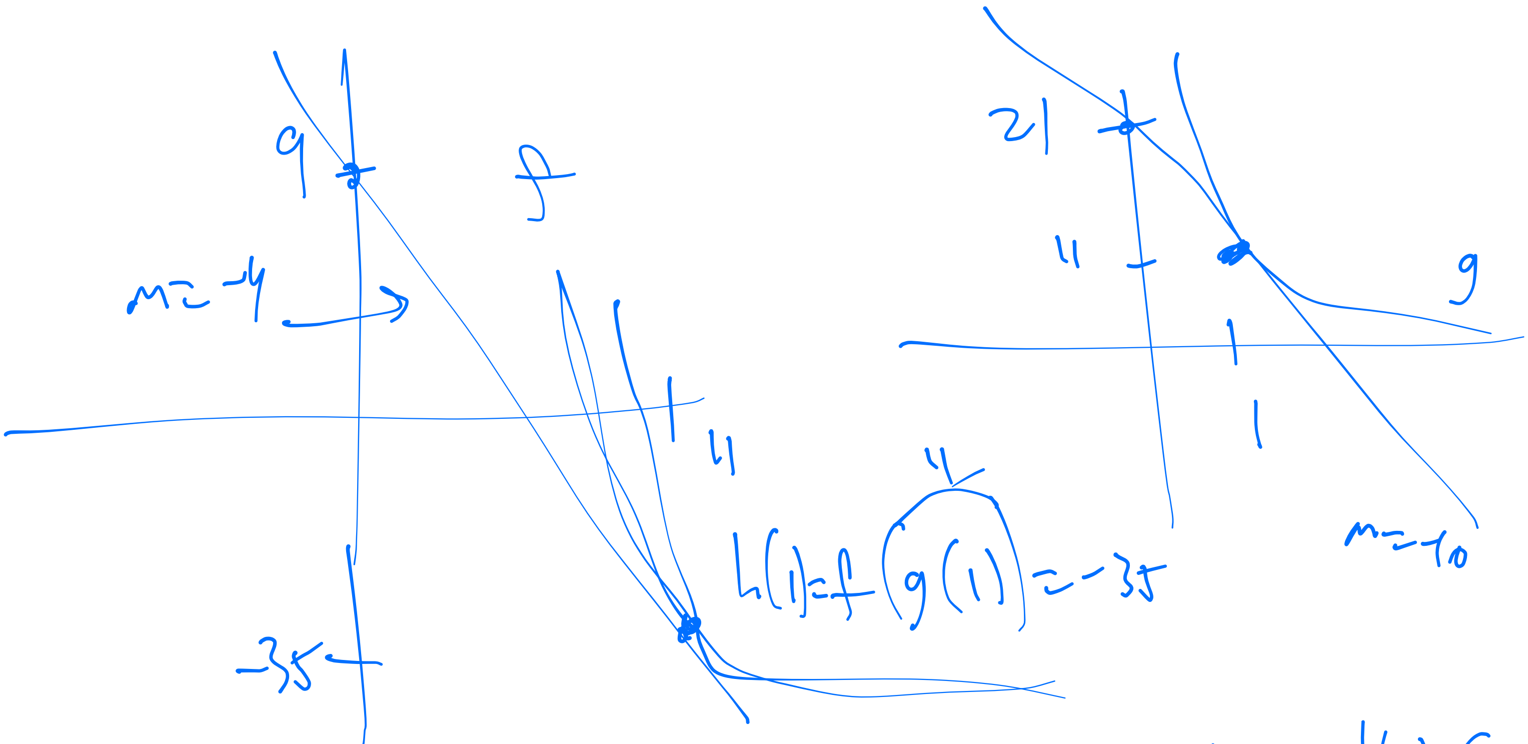


14. Let $f$ and $g$ be differentiable functions. Suppose the tangent line to the graph of $y = f(x)$ at $x = 11$ is $y = -4x + 9$, and the tangent line to the graph of $y = g(x)$ at $x = 1$ is $y = -10x + 21$. Use this information in parts (a) and (b).

(a) Let $h(x) = (f \circ g)(x)$. Find $h'(1)$ and an equation of the tangent line to the graph of $y = h(x)$ at $x = 1$.

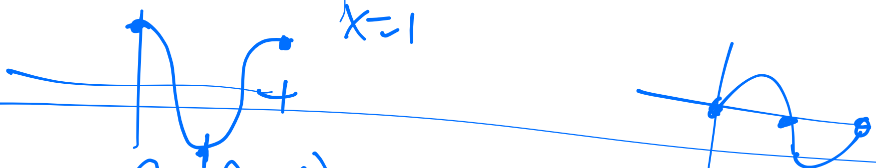
$y + 35 = 40(x-1)$.

(b) Let $p(x) = 6x\sin(\pi f(x))$. Find $p'(11)$ and an equation of the tangent line to the graph of $y = p(x)$ at $x = 11$.




$h(1) = f(g(1)) = -35$

$$h'(x)\Big|_{x=1} = f'(g(x))\cdot g'(x)\Big|_{x=1} = f'(g(1))\cdot g'(1) = f'(11)(-10) = (40).$$



$$p = 6x\cdot \sin(\pi f(x)), \quad p(11) = \underbrace{66\cdot \sin(\pi\cdot(-35))}_{=0}$$

$$p' = 6\left[1\cdot \sin(\pi f(x)) + x\cdot \cos(\pi f(x))\cdot \pi\cdot f'(x)\right]$$

$$x=11 \quad = 6\left[0 + 11\cdot \underbrace{\cos(\pi(-35))}_{-1}\cdot \pi\cdot(-4)\right].$$

$= 264\pi.$

15. Find $f'(3)$ if $f(g(x)) = e^{x^2}$, $g(1) = 3$, and $g'(1) = 6$.

$$\boxed{y - 0 = 264\pi(x - 11)}$$

$$f'(g(x)) \cdot g'(x) \Big|_{x=1} = e^{x^2}(2x) \Big|_{x=1}$$

$$f'(3) \cdot 6 = 2e$$

16. Consider the curve with equation $x^{5y} = y^{6x}$. Use this equation in parts (a) and (b).

(a) Use implicit differentiation to find $\dfrac{dy}{dx}$.

(b) Find an equation of the tangent line to the curve at the point $(1, 1)$.

$$5y \ln x = 6x \ln y$$

17. Suppose that $f(x)$ is a differentiable function such that $f(0) = \frac{1}{3}$ and $f'(0) = 347$. Evaluate the following derivatives. (Hint: Start by applying the chain rule.)

(a) $\left.\frac{d}{dx}\left[\cos(\sin^{-1}(f(x)))\right]\right|_{x=0}$ (b) $\left.\frac{d}{dx}\left[\sin(\tan^{-1}(f(x)))\right]\right|_{x=0}$ (c) $\left.\frac{d}{dx}\left[\cot(\sec^{-1}(\sqrt{1+f(x)}))\right]\right|_{x=0}$

18. Find the value of $b$ if

$b = e^7$.

(a) $f(x) = \cos(18x)b^x$ and $f'(0) = 7$.

(b) $f(x) = \sin(18x)b^{x+1}$ and $f'(0) = 7$.

(c) $f(x) = \dfrac{\log_b(x+1)}{\cos(18x)}$ and $f'(0) = 7$.

(d) $f(x) = \sin(\log_b(18x+1))$ and $f'(0) = 7$.

$$7 = f'(x)\Big|_{x=0} = (-\cancel{\sin(18x)})(18)\cdot b^x + \cancel{\cos(18x)}^{\,1}\cdot \cancel{b^x}\ln b\Big|_{x=0}$$

$$= \ln b$$

$$f' = 3x^2 + 10$$

19. Let $f(x) = x^3 + 10x + 8$. Use $f$ in parts (a)-(c).

(a) Find $f'(x)$.

(b) Find $(f^{-1})'(19)$.

(c) Find an equation of the tangent line to $y = f^{-1}(x)$ when $x = 19$.

$$\frac{1}{f'(\underbrace{f^{-1}(19)}_{1})} = \frac{1}{13}.$$

$$x^3 + 10x + \cancel{8} = 1\cancel{9}$$

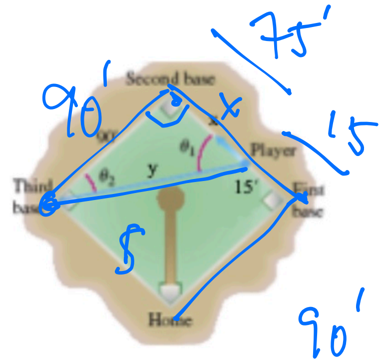
20. A rope is attached to the bottom of a hot-air balloon that is floating above a flat field. If the angle of the rope to the ground remains $\frac{\pi}{6}$ radians and the rope is pulled in at 3 ft/sec, how quickly is the elevation of the balloon decreasing when the balloon is 33 ft above the ground?

Be sure you define any variables used, draw an appropriate picture, use calculus to justify your answer, and report your answer in a sentence with appropriate units.

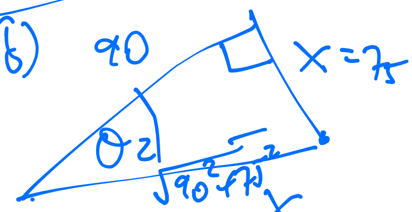
21. A spherical iron ball 12 in. in diameter is coated with a layer of ice of uniform thickness. If the ice melts at a rate of 14 $\text{in}^3/\text{min}$, how fast is the thickness of the ice decreasing when it is 4 in thick? How fast is the outer surface area of the ice decreasing?

Be sure you define any variables used, draw an appropriate picture, use calculus to justify your answer, and report your answer in a sentence with appropriate units.

22. A baseball diamond is a square 90 ft on a side. A player runs from first base to second at a rate of 18 ft/sec.




(b)




$\tan\theta_2 = \dfrac{x}{90}$

(a) At what rate is the player's distance from third base changing when the player is 15 ft from first base?

$-18 \cdot 75 \,/\, \sqrt{90^2+75^2}$

(b) At what rates are the angles $\theta_1$ and $\theta_2$ (see the figure) changing at that time?

(c) The player slides into second base at the rate of 15 ft/sec. At what rates are the angles $\theta_1$ and $\theta_2$ changing as the player touches second base?

$x = 90 \tan\theta_2$

$\dfrac{dx}{dt} = 90 \sec^2\theta_2 \dfrac{d\theta_2}{dt} = \dfrac{90^2+75^2}{90} \dfrac{d\theta_2}{dt}$

$x$ = distance runner to second

$\dfrac{dx}{dt} = -18 \dfrac{ft}{s}.$

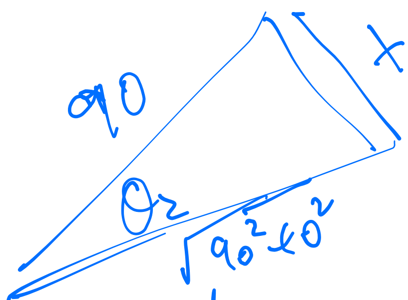
$90^2 + x^2 = s^2$

$s$ = distance to third

$\cancel{2}x \cdot \dfrac{dx}{dt} = \cancel{2}s \dfrac{ds}{dt}$

$= 75(-18) = \sqrt{90^2+75^2} \dfrac{ds}{dt}$

(c)




$x \to 0$

$\dfrac{dx}{dt} = -15 \, ft/s.$

$-15 = \dfrac{dx}{dt} = 90 \cancel{\sec^2\theta_2} \dfrac{d\theta_2}{dt}$ 1

$\boxed{\dfrac{d\theta_2}{dt} = \dfrac{-15}{90} \dfrac{rad}{sec}}$