Recall: $Q = \{A, B, C\} = Ax^2 + Bxy + Cy^2$

What $n = Q(x,y)$? Last time: if

$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \underbrace{GL_2(\mathbb{Z})}_{\text{"general linear"}}$ then $Q_1 = Q \circ \gamma$ represents the same n as Q.

What are the coefficients of Q_1?

$$Q_1(x,y) = Q\Big(\cancel{(x,y)\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}}_{\begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}} \Big).$$

$$= Q(ax+by,\ cx+dy).$$

$$= A(ax+by)^2 + B(ax+by)(cx+dy) + C(cx+dy)^2$$

$$= A_1 \cdot x^2 + B_1 \cdot xy + C_1 \cdot y^2,$$

where

$$A_1 = A \cdot a^2 + Bac + Cc^2 = Q(a,c).$$

$$B_1 = A \cdot 2ab + B(ad+bc) + C \cdot 2cd.$$

$$C_1 = A \cdot b^2 + B \cdot bd + Cd^2 = Q(b,d).$$

Last time: There is a duality between fixing Q & changing x & y, & fixing $(x \& y) = (1,0)$ & varying Q!

Ex: $Q_0 = 7x^2 - 11xy + 5y^2$

$Q(0,1)=5$ \ $Q_0(1,0)=7$	1	2	3	4	5	6
1	1	11	35	73	125	191
2	5		17		85	
3	19	7		25	55	
4	43		11		35	
5	77	43	23	17		47
6	121				25	

$Q_0(1,1)$

$\overset{\shortparallel}{(x^2 - xy + 5y^2)}\Big|_{(1,0)}$

$[23, 21, 5]\big|_{(1,0)}$

Start with $Q_0 = [7, -11, 5]$. & $Q_0(1,0) = 7$.

Idea: Change Q_0 to new Q & evaluate all $(1,0)$.

Want: $Q(1,0) = Q_0(1,1) = Q_0\Big[\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}\Big]$.

Need to solve: $\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{= \begin{pmatrix} a \\ c \end{pmatrix}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. ← arbitrary.

Condition:

$\overset{\displaystyle ad - bc}{\shortparallel} = 1.$

$1 \cdot d - b = 1.$

So $Q = Q_0 \circ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = [7, -11, 5] \circ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$.

$= [A_1, B_1, C_1] = [1, 0 + (-11) + 10, 5]$

(2) $= [1, -1, 5]$.

Ex: $Q_0 = [7, -11, 5]$, $Q_0(3,5) = 23$.

Give me one $Q = Q_0 \circ \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ s.t. $Q(1,0) = Q_0(3,5)$.

Want: $Q(1,0) = Q_0\left(\underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\begin{pmatrix} 3 \\ 5 \end{pmatrix}}\right)$. $Q = Q_0 \circ \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$

$\Rightarrow \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{\begin{pmatrix} a \\ c \end{pmatrix}} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}$.

$\Rightarrow \left|\begin{pmatrix} 3 & b \\ 5 & d \end{pmatrix}\right| = 3 \cdot d - 5 \cdot b = 1$.

Euclidean alg! $b = 1$, $d = 2$. ✓

$Q = [7, -11, 5] \circ \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix} = [23, \underbrace{42 - 121 + 100}_{21}, 5]$

$Q = 23x^2 + 21xy + 5y^2$, $= Q_0 \circ \begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$, $Q(1,0) = 23$.

Def: $Q_1 \sim Q_2$ "equivalent" if $\exists\, \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$

s.t. $Q_1 = Q_2 \circ \gamma$.

"special linear"

If $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\gamma^{-1} = \frac{1}{\det}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

det = ad − bc $\qquad \gamma \circ \gamma^{-1} = \frac{1}{\det}\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} \cancel{ad-bc}^{\,1} & 0 \\ 0 & \cancel{ad-bc}^{\,1} \end{pmatrix}$.

So $GL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, \frac{1}{\det} \in \mathbb{Z} \Leftrightarrow \det = \pm 1 \right\}$.

$\cup$

$SL_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, \det = +1 \right\}$.

Q: Is "$\sim$" an equivalence relation?

(i) reflexive: $Q \sim Q = Q \circ \begin{pmatrix}1&0\\0&1\end{pmatrix}$.

(ii) symmetric $Q_1 = Q_2 \circ \begin{pmatrix}a&b\\c&d\end{pmatrix}$, $\boxed{\text{Quiz 1: } \cancel{\text{Exercise}}}$ $Q_2 = Q_1 \circ \boxed{???}$ inverse

(iii) transitive $Q_1 = Q_2 \circ \begin{pmatrix}a&b\\c&d\end{pmatrix}$, $Q_2 = Q_3 \circ \begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix}$

$\boxed{\text{Quiz 2:}}$ $\Rightarrow Q_1 = Q_3 \circ \boxed{???}$ matrix multiplication.

$\boxed{\text{Quiz 3:}}$ Find $Q = \underbrace{Q_0}_{[7,-11,5]} \circ \begin{pmatrix}a&b\\c&d\end{pmatrix}$ s.t.

$$Q(1,0) = Q_0(5,2) = 85$$

Lemma: If $Q_1 \sim Q_2$ then

$$\{n \in \mathbb{Z} \mid \exists x,y \in \mathbb{Z},\ Q_1(x,y) = n\} = \{n \in \mathbb{Z} \mid \exists x,y \in \mathbb{Z},\ Q_2(x,y) = n\}.$$

i.e. Equivalent forms represent the same numbers.

This lemma means: If we had started with $Q = [1, -1, 5]$ instead of $Q_0 = [7, -11, 5]$ & made tables of its values, we would see the same numbers. $1, 5, 11, 17, 7, 35, \ldots$ just in different locations (shuffled around).

$Q = [1, -1, 5]$

$Q(1,0) = 1$

$Q(0,1) = 5$

	1	2	3	4	5
1	5	7	11	17	
2			23		
3		43			
4	77		77		
5	121		113		

$Q(3,5) = 3 - 15 + 125$
$= 113.$

<u>Exercise</u> Fill in

<u>pf of Lemma:</u> If $n = Q_1(x,y)$ then $(Q_2 = Q_1 \circ \gamma)$.

$$\begin{aligned} n &= Q_2(x_1, y_1), \\ &= Q_1\left(\gamma\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right) \\ &= Q_1\left(\gamma \cdot \gamma^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right) = Q_1(x,y). \end{aligned}$$

<u>where</u>

$\gamma \cdot \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$

$\Rightarrow \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \gamma^{-1}\begin{pmatrix} x \\ y \end{pmatrix}.$

<u>Non-E.g.</u>: $\gamma = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ invertible? det $= 2$.

$\gamma^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}.$ Not integer matrix.

$Q_1 = (\underbrace{x^2 + y^2}_{Q_0}) \circ \gamma$, $\quad [1, 0, 1] \circ \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = [4, 0, 1]$,

$= (2x)^2 + y^2 = 4x^2 + y^2.$

Does Q_1 represent same numbers as $Q_0 = x^2 + y^2$?

<u>Obs:</u> $2 = 1^2 + 1^2$. rep by Q_0, but $2 \neq 4x^2 + y^2$.