Where are we? Know: solve $x^2+y^2=n$, & $x^2-xy+y^2=n$. (When there are / aren't solutions, & how to find prime solutions efficiently).

General question: How about general binary (i.e. two variable, x & y) quadratic form.

$$Q(x,y) = Ax^2+Bxy+Cy^2 = [A,B,C].$$

E.g.: $x^2+y^2=[1,0,1]$, $D=-4$; $x^2-xy+y^2=[1,-1,1]$, $D=-3$

Non-e.g.: $Q(x,y) = x^2+2xy+y^2 = [1,2,1]$.

$= (x+y)^2$. For which n is $Q(x,y)=n$ solvable? Replace x with $x-y$.

After change of variables $\underbrace{z=x+y,\ w=y}$ invertible $\mathbb{Z}$-linear change of variables.

$$Q_1(z,w)=z^2.$$

Only perfect squares are represented. This Q is degenerate: discriminant

$$D_Q = D = B^2-4AC = 0.$$

$$Q = x^2+2xy+y^2 \rightsquigarrow Q = x^2+2x+1. \quad (\text{set } y=1)$$
$$= (x+1)^2.$$

$y = Q(x)$.

discr $= 0$.

<u>Ex 2:</u> $Q = [1, -3, 2]$

$$= x^2 - 3xy + 2y^2.$$

$$(y \to 1) \to x^2 - 3x + 2 = (x-1)(x-2).$$

$$\to Q = (x-y)(x-2y).$$

$$D_Q = (-3)^2 - 4(1)(2) = 1.$$

What's "wrong" with this? $5 = (x-y)(x-2y)$?

$$5 = 1\cdot 5 = (-1)(-5) = 5\cdot 1 = (-5)\cdot(-1).$$

$$\begin{cases} x-y=1 \\ x-2y=5 \end{cases} \text{ or } \quad \vdots \qquad \begin{cases} x-y=-5 \\ x-2y=1 \end{cases}$$

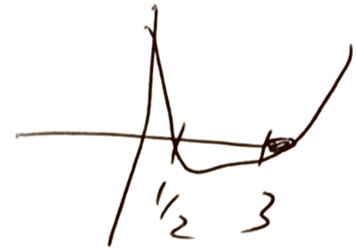
$$Q_1(\overset{x-y}{\overset{\shortparallel}{z}}, \overset{-y}{\overset{\shortparallel}{w}}) = z\cdot(z+w).$$

Set $z = 1$, $1\cdot(1+w) = 5$, $w = 4$.

$$Q_2 = \overset{z}{\overset{\shortparallel}{u}} \cdot \overset{z+w}{\overset{\shortparallel}{v}} = n \quad \text{solved by: } u=1, v=n.$$

Ex: $2x^2 - 7xy + 3y^2 = Q$

$y \to 1$ ⇝ $2x^2 - 7x + 3 = (2x-1)(x-3)$.



also splits (reducible). $D_Q = (-7)^2 - 4(2)(3)$
$= 25. = 5^2.$

Def: If Q has $D_Q = 0 \Rightarrow$ degenerate

If Q has $D_Q = d^2 \Rightarrow$ splits/reducible.

Telling whether a large # is a square is easy.

Ex: $\sqrt{1441}$

$30^2 \to 900$

$40^2 \to 1600$

$35^2 = 30^2 + 2\cdot 30 \cdot 5 + 5^2$
$= 1225.$

Ex: $Q = x^2 - 5xy + 3y^2$. Is Q definite or indefinite?

Def: Q is definite if its sign never changes.

$D_Q = 25 - 4(3) = 13 > 0$!!! Look at $(x^2 - 5x + 3)$
$= (x-\alpha)(x-\bar\alpha).$

$\alpha, \bar\alpha = \frac{5 \pm \sqrt{13}}{2}$.

So Q is ~~in~~ definite. E.g. $Q(0,1) = 3$

$Q(1,1) = -1.$

Q: What #s arise as discriminants?

$$D_Q = B^2 - 4AC$$

Lemma: $D_Q \equiv 0 \text{ or } 1 \bmod 4.$

pf: $D_Q \equiv B^2 \bmod 4,$ $\blacksquare$.

If Q splits & $D_Q = d^2$, $\alpha = \dfrac{-B \pm \sqrt{D}}{2A} = \dfrac{-B \pm d}{2A}.$

$\Rightarrow Q$ is a product of 2 linear polynomials $\in \mathbb{Q}$.

Exercise: Analyze $Q = 7x^2 - 2xy + y^2$

Degenerate?

Split?

Definite / Indefinite?

$D_Q = ?$
